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Renormalization and blow up for the critical Yang–Mills problem ☆

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Abstract

We consider the Yangs–Mills equations in $4+1$ dimensions. This is the energy critical case and we show that it admits a family of solutions which blow up in finite time. They are obtained by the spherically symmetric ansatz in the $SO(4)$ gauge group and result by rescaling of the instanton solution. The rescaling is done via a prescribed rate which in this case is a modification of the self-similar rate by a power of $|\log t|$. The powers themselves take any value exceeding $3/2$ and thus form a continuum of distinct rates leading to blow-up. The methods are related to the authors' previous work on wave maps and the energy critical semi-linear equation. However, in contrast to these equations, the linearized Yang–Mills operator (around an instanton) exhibits a zero energy eigenvalue rather than a resonance. This turns out to have far-reaching consequences, amongst which are a completely different family of rates leading to blow-up (logarithmic rather than polynomial corrections to the self-similar rate).

Keywords: Nonlinear wave equations; Blow up; Critical phenomena

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1. Introduction

We describe singularity formation for the semi-linear wave equation

$$\Box u - \frac{2}{r^2} u(1-u^2) = 0, \quad \Box = \partial_{tt} - \Delta \tag{1.1}$$

in $\mathbb{R}^{2+1}$. This equation arises as follows: consider Yang–Mills fields in $(d+1)$-dimensional Minkowski spacetime. The gauge potential A_α is a one-form with values in the Lie algebra $\mathfrak{g}$ of a compact Lie group G. In terms of the curvature $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta]$ the Yang–Mills equations take the form

$$\partial_\alpha F^{\alpha\beta} + \left[A_\alpha, F^{\alpha\beta}\right] = 0,$$

where $[\cdot,\cdot]$ is the Lie bracket on G. We take $G = SO(d)$ with $\mathfrak{g}$ being the skew-symmetric $d \times d$ matrices. In particular $A_\alpha = \{A_\alpha^{ij}\}_{i,j=1}^d$. Assuming the spherically symmetric ansatz (see [14] and [8] for analogous considerations in the context of the Yang–Mills heat flow)

$$A_\mu^{ij}(x) = \left(\delta_\mu^i x^j - \delta_\mu^j x^i\right) \frac{1-u(t,r)}{r^2}$$

the Yang–Mills equations reduce to the semi-linear wave equation

$$\Box_{d-2} u = \partial_{tt} u - \Delta_{d-2} u = \frac{d-2}{r^2} u\left(1-u^2\right).$$

This equation is invariant under the scaling $u(r,t) \mapsto u(r/\lambda, t/\lambda)$. With respect to this scaling the energy

$$E = \int_0^\infty \left[u_t^2 + u_r^2 + \frac{d-2}{2r^2}\left(1-u^2\right)^2\right] r^{d-3}\, dr$$

is invariant iff $d = 4$ which is the case we consider in this paper. Eq. (1.1) admits the stationary solution

$$Q(r) = \frac{1-r^2}{1+r^2},$$

called *instanton*. In $3+1$ dimensions, the Yang–Mills equations are subcritical relative to the energy. Eardley and Moncrief [6,7] showed that in that case there are global smooth solutions. See also Klainerman, Machedon [9] who lowered the regularity assumptions on the data. In the energy critical case of $4+1$ dimensions, local well-posedness in H^s with $s > 1$ was shown by Klainerman, Tataru [10]. However, it was conjectured that global well-posedness fails and that singularities should form, see Bizon, Tabor [1] and Bizon, Ovchinnikov, Sigal [2] for numerical and heuristic arguments to that effect. However, such a phenomenon had not been observed rigorously. In this paper we show how to construct a solution to the wave equation (1.1) as a perturbation of a time-dependent profile

$$u_0 = Q(R), \qquad R = r\lambda(t), \quad \Phi(R) = \frac{R^2}{(1+R^2)^2}$$

with $\lambda(t)$ a logarithmic correction to the self-similar ansatz

$$\lambda(t) = t^{-1}(-\log t)^{\beta}, \quad \beta > 0.$$

In other words, we prove that in general the energy critical Yang–Mills equations develop singularities in finite time.

As in our earlier work [11] for energy critical wave maps, and [12] for the energy critical semi-linear wave equation in $\mathbb{R}^3$, the blow up rate is prescribed. Since a continuum of rates is admissible, the blow up solutions which we construct can of course not be stable. In contrast to the rates $\lambda(t) = t^{-1-\nu}$ which appeared in [11] and [12], in the case of Yang–Mills we only make logarithmic corrections to the self-similar rate. This has to do with the fact that the linearized Yang–Mills operator has a zero energy eigenvalue in $4+1$ dimensions, whereas for wavemaps as well as the three-dimensional semi-linear focusing wave equation, it exhibits a zero energy resonance. This difference is very important as in the case of an eigenvalue an orthogonality condition appears which is not present in the zero energy resonant case. It is this fact which required major changes to our scheme, especially to the "renormalization part" in which we construct approximate solutions. In addition, in contrast to our earlier work on wave maps [11], the approximate solutions here are much rougher, and indeed asymptotically only lie in H^1, the threshold for local well-posedness of the critical Yang–Mills equation. The reason for this is the much more singular nature of the ODE's arising in the renormalization step, due to the different blow up rate.

Theorem 1.1. *Let*

$$\lambda(t) := t^{-1}(-\log t)^{\beta}.$$

For each $\beta > \frac{3}{2}$ there exists a spherically symmetric solution u to (1.1) *inside the cone* $\{r < t,\ t < t_0\}$ *which has the form*

$$u(x,t) = Q\big(r\lambda(t)\big) + v(x,t)$$

where the function v has the size and regularity, with $S := t\partial_t + r\partial_r$,

$$\|\nabla v\|_{L^2} + \|\nabla S v\|_{L^2} + \big\|\nabla S^2 v\big\|_{L^2} \lesssim |\log t|^{-1}$$

as well as the pointwise decay

$$\big|v(t,x)\big| \lesssim |\log t|^{-1}.$$

We emphasize that our solutions are just barely better than H^1, in contrast to our earlier work on wavemaps. While H^1 local well-posedness is not known for the general Yang–Mills problem, it is known in the equivariant case, see [15]. This is important for our purposes, as it shows that our solutions belong to a class for which a local well-posedness theory is available. In addition, the vector field S is required to control the strong singularity in the nonlinearity $r^{-2}u^3$ at $r=0$; this is in the spirit of the method of invariant vector fields in nonlinear wave equations which allows for improved decay away from the characteristic light-cone $\{|x|=t\}$. More precisely, one can use elliptic estimates close to $r=0$ to control the aforementioned singularity. For related work in this area see [3], [4], and [13].

2. The proof of the main theorem

This section contains the proof of Theorem 1.1. The first step is construct an arbitrarily good approximate solution to the wave equation (1.1) as a perturbation of a time-dependent profile $u_0 = Q(R)$. The result is as follows:

Theorem 2.1. *For each integer N there exists a spherically symmetric approximate solution u_N to* (1.1) *inside the cone $\{r<t,\ t<t_0\}$ which has the form*

$$u_N(r,t) = Q(R) + v_{10}(r,t) + v_N(r,t), \quad v_{10}(r,t) := \big(t\lambda(t)\big)^{-2}\frac{R^4}{4(1+R^2)^2},$$

with $\lambda(t) = t^{-1}|\log t|^{\beta}$, $R = r\lambda(t)$, and v_N satisfying the pointwise bounds

$$\big|v_N(r,t)\big| + \big|Sv_N(r,t)\big| + \big|S^2 v_N(r,t)\big| \lesssim \frac{r^2}{t^2|\log t|} = \frac{R^2}{(t\lambda(t))^2|\log t|}$$

and so that the corresponding error

$$e_N = \Box u_N - \frac{2}{r^2}u_N\big(1-u_N^2\big)$$

satisfies

$$\big|e_N(r,t)\big| + \big|Se_N(r,t)\big| + \big|S^2 e_N(r,t)\big| \lesssim t^{-2}|\log t|^{-N}.$$

The proof of the above theorem is carried out in Section 3. The description of the approximate solutions u_N obtained there is much more precise than what is stated above. In particular, the functions u_N are analytic up to the cone $t=r$, and the nature of the singularity at the cone is clearly explained.

Given the approximate solutions u_N constructed above, we look for a solution u to (1.1) of the form

$$u(t,r) = u_N(t,r) + \varepsilon(t,r),$$

where ε is to be determined via Banach iteration. The equation for ε is

$$\left(-\partial_t^2 + \partial_r^2 + \frac{1}{r}\partial_r + \frac{2}{r^2}\big(1 - 3u_N(t,r)^2 - 3\varepsilon(t,r)u_N(t,r) - \varepsilon^2(t,r)\big)\right)\varepsilon(t,r) = e_N.$$

We divide this equation into a linear part and a nonlinear perturbative term. Based on past experience one would expect that in the main linear part u_N is simply replaced by $Q(\lambda(t)r)$. However, in this case that is not enough. Instead, as it turns out, some of the effects of the first correction term v_{10} also need to be taken into account. Hence the above equation is rewritten in the form

$$\left(-\partial_t^2+\partial_r^2+\frac{1}{r}\partial_r+\frac{2}{r^2}\bigl(1-3Q\bigl(\lambda(t)r\bigr)^2-6Q\bigl(\lambda(t)r\bigr)v_{10}\bigr)\right)\varepsilon=e_N+\mathcal{N}(\varepsilon) \tag{2.1}$$

where

$$\mathcal{N}(\varepsilon)=\frac{2}{r^2}\bigl(3\varepsilon\bigl(u_N^2-Q\bigl(\lambda(t)r\bigr)^2-2Q\bigl(\lambda(t)r\bigr)v_{10}\bigr)+3\varepsilon^2u_N+\varepsilon^3\bigr).$$

We first consider the linear problem

$$\left(-\partial_t^2+\partial_r^2+\frac{1}{r}\partial_r+\frac{2}{r^2}\bigl(1-3Q\bigl(\lambda(t)r\bigr)^2-6Q\bigl(\lambda(t)r\bigr)v_{10}\bigr)\right)\varepsilon=f \tag{2.2}$$

where the principal spatial part is given by the selfadjoint operator

$$L_t=-\partial_r^2-\frac{1}{r}\partial_r-\frac{2}{r^2}\bigl(1-3Q\bigl(\lambda(t)r\bigr)^2\bigr).$$

This is time dependent, but is obtained by rescaling from the operator

$$L=-\partial_r^2-\frac{1}{r}\partial_r-\frac{2}{r^2}\bigl(1-3Q(r)^2\bigr).$$

We remark that, as proved in the next section, L is a nonnegative operator.

A difficulty that we face in solving (2.1) iteratively is in handling the singularity at 0 in the ε^3 term in $\mathcal{N}(\varepsilon)$. Energy estimates on ε do not suffice, so we introduce the scaling vector field

$$S:=t\partial_t+r\partial_r$$

and we seek to simultaneously bound ε, $S\varepsilon$ and $S^2\varepsilon$ in a norm that is a scale adapted version of the H^1 norm,

$$\|\varepsilon\|_{H_N^1}:=\sup_{0<t<t_0}|\log t|^{N-\beta-1}\bigl(\|L_t^{\frac{1}{2}}\varepsilon\|_{L^2(r\,dr)}+\|\partial_t\varepsilon\|_{L^2(r\,dr)}+\lambda(t)|\log t|^{-\beta}\|\varepsilon\|_{L^2(r\,dr)}\bigr).$$

For f, on the other hand, we just use uniform L^2 bounds,

$$\|f\|_{L_N^2}:=\sup_{0<t<t_0}\lambda^{-1}(t)|\log t|^N\bigl\|f(t)\bigr\|_{L^2(r\,dr)}.$$

The main result of the linear theory is the following theorem. It is proved in Section 6.

Theorem 2.2. *There exists a linear operator Φ, so that for each f the function $\varepsilon=\Phi f$ solves* (2.2)*, and for all large enough $N_0\gg N_1\gg N_2$ it satisfies the bounds*

$$\|\Phi f\|_{H^1_{N_0}} \lesssim \frac{1}{N_0}\|f\|_{L^2_{N_0}}, \tag{2.3}$$

$$\|S\Phi f\|_{H^1_{N_1}} \lesssim \frac{1}{N_1}\big(\|Sf\|_{L^2_{N_1}} + \|f\|_{L^2_{N_0}}\big), \tag{2.4}$$

$$\big\|S^2\Phi f\big\|_{H^1_{N_2}} \lesssim \frac{1}{N_2}\big(\big\|S^2 f\big\|_{L^2_{N_2}} + \|Sf\|_{L^2_{N_1}} + \|f\|_{L^2_{N_0}}\big). \tag{2.5}$$

The implicit constants here depend only on β.

We note that Φ is in effect the forward solution operator for Eq. (2.2). In this theorem f is not required to be supported inside the cone $\{r \leqslant t\}$. However, if that is the case the Φf is also supported inside the cone due to the finite speed of propagation.

In order to prove the above theorem it is convenient to pass to different coordinates in which the Schrödinger operator is no longer time dependent. Specifically, introduce new coordinates (τ, R) given by

$$\tau = \int_t^{t_0} \lambda(s)\,ds, \qquad R = \lambda(t)r.$$

Then, denoting

$$\tilde{\varepsilon}(\tau, R) := R^{\frac{1}{2}}\varepsilon(t, r), \qquad \tilde{f}(\tau, R) := \lambda^{-2}R^{\frac{1}{2}}f(t, r)$$

where λ is now understood as a function of τ, Eq. (2.2) becomes

$$\left[-\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)^2 + \frac{1}{4}\left(\frac{\lambda_\tau}{\lambda}\right)^2 + \frac{1}{2}\partial_\tau\left(\frac{\lambda_\tau}{\lambda}\right)\right]\tilde{\varepsilon} - \mathcal{L}\tilde{\varepsilon} - \frac{12}{R^2}Q(R)v_{10}\,\tilde{\varepsilon} = \tilde{f},$$

where

$$\mathcal{L} := -\frac{\partial^2}{\partial R^2} + \frac{5}{14R^2} - \frac{24}{(1+R^2)^2}.$$

The spectral properties of the operator $\mathcal{L}$ are studied in Section 4. These are essential in the proof of Theorem 2.2 in Section 6.

We next continue the proof of our main result with the perturbative argument for Eq. (2.1). By the construction in Section 3 we know that for arbitrarily large $N_0 \gg N_1 \gg N_2$ we can find an approximate solution u_N so that the corresponding error e_N satisfies

$$\|e_N\|_Y := \|e_N\|_{L^2_{N_0}} + \|Se_N\|_{L^2_{N_1}} + \big\|S^2 e_N\big\|_{L^2_{N_2}} \ll 1$$

where the smallness is gained by taking t_0 small enough. It is important to note that, even though e_N has limited regularity, the roughness is relative to the self-similar variable $a := \frac{r}{t}$ which satisfies $Sa = 0$. For this reason $S^j e_N$ does not lose any regularity. We iteratively construct the sequence $\{(\varepsilon_j, f_j)\}_{j \geqslant 0}$ by

$$f_0 := e_N, \qquad \varepsilon_j := \Phi f_j, \qquad f_{j+1} := e_N + \mathcal{N}(\varepsilon_j)$$

and show that it converges to a solution ε of (2.1) in the norm

$$\|\varepsilon\|_X := \|\varepsilon\|_{H^1_{N_0}} + \|S\varepsilon\|_{H^1_{N_1}} + \left\|S^2\varepsilon\right\|_{H^1_{N_2}}$$

By Theorem 2.2 we know that Φ is a bounded operator with small norm,

$$\|\Phi f\|_X \lesssim N_0^{-1}\|f\|_Y. \tag{2.6}$$

The proof is concluded if we show that the nonlinear term satisfies a similar bound:

Lemma 2.3. *The map $f \mapsto \mathcal{N}(\Phi f)$ is locally Lipschitz from Y to Y, with Lipschitz constant of size $O(N_2^{-1})$.*

Proof. We denote $\varepsilon = \Phi f$, and successively consider the linear and the nonlinear terms in $\mathcal{N}(\varepsilon)$.

A. *The linear term* has the form

$$\mathcal{N}_1(\varepsilon) = g\varepsilon, \qquad g = \frac{2}{r^2}\big(u_N^2 - Q\big(\lambda(t)r\big)^2 - 2Q\big(\lambda(t)r\big)v_{10}\big).$$

By construction we have

$$|g| + |Sg| + \left|S^2 g\right| \lesssim \frac{1}{t^2|\log t|} = \lambda(t)^2|\log t|^{-2\beta-1}$$

which directly leads to

$$\left\|\mathcal{N}_1(\varepsilon)\right\|_Y \lesssim \|\varepsilon\|_X$$

where only the L^2 components of the $H^1_{N_j}$ norms are being used (as part of the space X). The desired conclusion now follows from (2.6).

B. *The nonlinear term* has the form

$$\mathcal{N}_2(\varepsilon) = \frac{2}{r^2}\big(3u_N\varepsilon^2 + \varepsilon^3\big).$$

The coefficient u_N satisfies

$$|u_N| + |Su_N| + \left|S^2 u_N\right| \lesssim 1$$

so we can neglect it. The main difficulty here arises from the singular factor $\frac{1}{r^2}$ on the right-hand side. To address that we will establish several bounds. The first is a pointwise bound,

$$|w| \lesssim |\log t|^{1+2\beta-N}\|w\|_{H^1_N} \tag{2.7}$$

which applied to ε, $S\varepsilon$ and $S^2\varepsilon$ yields

$$\left\|S^k\varepsilon(t)\right\|_{L^\infty} \lesssim \frac{1}{N_k}|\log t|^{1+2\beta-N_k}\|f\|_Y, \quad k = 0, 1, 2. \tag{2.8}$$

The second is a weighted L^2 bound, namely

$$\left\| r^{-2}\varepsilon(t)\right\|_{L^2} \lesssim \lambda(t)|\log t|^{-N_2+1}\|f\|_Y. \tag{2.9}$$

Interpolating between the $k=2$ case of (2.8) and (2.9) we also obtain

$$\left\| r^{-1}S\varepsilon(t)\right\|_{L^4} \lesssim \frac{1}{N_2^{\frac{1}{2}}}\lambda(t)^{\frac{1}{2}}|\log t|^{-N_2+1+\beta}\|f\|_Y. \tag{2.10}$$

By (2.8) with $k=0$ and (2.9) we obtain

$$\begin{aligned}\left\|\mathcal{N}_1(\varepsilon)(t)\right\|_{L^2_{N_0}} &\lesssim \left[\left\|\varepsilon(t)\right\|_{L^\infty} + \left\|\varepsilon(t)\right\|^2_{L^\infty}\right]\left\|\frac{\varepsilon(t)}{r^2}\right\|_{L^2} \\ &\lesssim \frac{1}{N_0}\lambda(t)|\log t|^{2+2\beta-N_0-N_2}\left(\|f\|_Y^2 + \|f\|_{Y^3}\right).\end{aligned} \tag{2.11}$$

Using also (2.8) with $k=1$ we similarly obtain

$$\left\| S\mathcal{N}_1(\varepsilon)(t)\right\|_{L^2_{N_0}} \lesssim \frac{1}{N_1}\lambda(t)|\log t|^{2+2\beta-N_1-N_2}\left(\|f\|_Y^2 + \|f\|_{Y^3}\right). \tag{2.12}$$

Finally, due to (2.8) with $k=2$ and (2.10) we also have

$$\left\| S^2\mathcal{N}_1(\varepsilon)(t)\right\|_{L^2_{N_0}} \lesssim \frac{1}{N_2}\lambda(t)|\log t|^{2+2\beta-2N_2}\left(\|f\|_Y^2 + \|f\|_{Y^3}\right). \tag{2.13}$$

Together, the bounds (2.11)–(2.13) suffice to obtain the conclusion of the lemma provided that N_2 is large enough.

It remains to prove the bounds (2.7) and (2.9). For the operator L we have the straightforward elliptic bound

$$\|\nabla w\|_{L^2} + \|r^{-1}w\|_{L^2} \lesssim \left\| L^{\frac{1}{2}}w\right\|_{L^2} + \|w\|_{L^2}.$$

By rescaling this gives

$$\begin{aligned}\|\nabla w\|_{L^2} + \left\|r^{-1}w\right\|_{L^2} &\lesssim \left\| L_t^{\frac{1}{2}}w\right\|_{L^2} + \lambda(t)\|w\|_{L^2} \\ &\lesssim |\log t|^{1+2\beta-N}\|w\|_{H^1_N}.\end{aligned} \tag{2.14}$$

Then (2.7) follows from the pointwise bound for spherically symmetric functions in $\mathbb{R}^2$

$$\|u\|_{L^\infty} \lesssim \|\nabla u\|_{L^2} + \left\|r^{-1}u\right\|_{L^2}.$$

Next we turn our attention to the bound (2.9). Due to (2.8) ($k=0$) it suffices to consider the region $r \leqslant t/2$. In this region we use the scaling vector field $S = t\partial_t + r\partial_r$ to derive a stronger equation for ε. From

$$t\partial_t = S - r\partial_r$$

one infers that

$$t^2\partial_t^2\varepsilon + t\partial_t\varepsilon = -S^2\varepsilon + r^2\partial_r^2\varepsilon + 2t\,\partial_t S\varepsilon$$

and further

$$\left(\frac{t^2-r^2}{t^2}\partial_{rr} + \frac{1}{r}\partial_r - \frac{4}{r^2}\right)\varepsilon = t^{-2}\left(-S^2\varepsilon + 2t\,\partial_t S\varepsilon - t\,\partial_t\varepsilon\right) - \left(\Box + \frac{4}{r^2}\right)\varepsilon.$$

Due to (2.2) we can estimate the last term by

$$\left|\left(\Box + \frac{4}{r^2}\right)\varepsilon\right| \lesssim |f| + \lambda^2|\varepsilon|.$$

This leads to the bound

$$\begin{aligned}
&\left\|\left(\frac{t^2-r^2}{t^2}\partial_{rr} + \frac{1}{r}\partial_r - \frac{4}{r^2}\right)\varepsilon\right\|_{L^2} \\
&\quad\lesssim t^{-2}\left[\left\|S^2\varepsilon\right\|_{L^2} + \left\|(t\partial_t S)\varepsilon\right\|_{L^2} + \left\|(t\partial_t)\varepsilon\right\|_{L^2}\right] + \lambda^2\|\varepsilon\|_{L^2} + \|f\|_{L^2} \\
&\quad\lesssim \lambda(t)|\log t|^{1-N_2}\|\varepsilon\|_X + \|f\|_{L^2}.
\end{aligned}$$

Taking also into account (2.14) applied to ε with $N = N_0$, the estimate (2.9) would follow from the fixed time bound

$$\left\|r^{-2}\varepsilon\right\|_{L^2} \lesssim \left\|t^{-1}\nabla\varepsilon\right\|_{L^2} + \left\|t^{-1}r^{-1}\varepsilon\right\|_{L^2} + \left\|\left(\frac{t^2-r^2}{t^2}\partial_{rr} + \frac{1}{r}\partial_r - \frac{4}{r^2}\right)\varepsilon\right\|_{L^2}. \tag{2.15}$$

This rescales to $t = 1$, in which case it is a standard local elliptic estimate near $r = 0$. $\Box$

3. The renormalization step

In this section, roughly following [11], we show how to construct an arbitrarily good approximate solution to the wave equation (1.1) as a perturbation of a time-dependent profile

$$u_0 = Q(R), \qquad R = r\lambda(t), \quad \Phi(R) = \frac{R^2}{(1+R^2)^2} \tag{3.1}$$

with $\lambda(t)$ a logarithmic correction to the self-similar ansatz

$$\lambda(t) = t^{-1}(-\log t)^{\beta}, \quad \beta \geqslant 1.$$

In fact, for ease of notation we will take $\beta \in \mathbb{Z}$; the general case is only a minor modification of the integral one and we leave it to the reader. This ansatz is quite natural in light of a necessary orthogonality condition which makes its appearance in the ensuing considerations. We note, however, that by contrast to [11], the approximate solutions here are much rougher, and indeed asymptotically only lie in H^1, the threshold for local well-posedness of the critical Yang–Mills equation. The reason for this is the much more singular nature of the ODE's arising in the renormalization step, due to the different blow up rate.

The following is the main theorem of the first half of the paper. Throughout this section, we will work on the light-cone $\{r < t\}$ (in particular, all functions in this section will be defined only on $r \leqslant t$).

Theorem 3.1. *Let $k \in \mathbb{N}$. There exists an approximate solution $u_{2k-1} \in H^1$ for* (1.1) *of the asymptotic form (as $R \to \infty$)*

$$u_{2k-1}(r,t) = Q\big(\lambda(t)r\big) + \frac{1}{(t\lambda)^2}\frac{R^4}{4(1+R^2)^2} + O\left(\frac{R^2}{(t\lambda)^2|\log t|}\right)$$

so that the corresponding error has size

$$e_{2k-1} = O\left(\frac{1}{t^2(t\lambda)^{2k}}\right).$$

Here the $O(\cdot)$ terms are uniform in $0 \leqslant r \leqslant t$ and $0 < t < t_0$ where t_0 is a fixed small constant; they are also stable with respect to the application of powers of the scaling operator S. We also have $u_{2k-1}(\cdot,t) \in C^\infty([0,t))$, and further $u_{2k-1} \in H^1$. The only singularity arises on the light cone $r = t$.

Proof. We iteratively construct a sequence u_k of better approximate solutions by adding corrections v_k,

$$u_k = v_k + u_{k-1}.$$

The error at step k is

$$e_k = \left(\partial_t^2 - \partial_r^2 - \frac{1}{r}\partial_r\right)u_k - \frac{2}{r^2}f(u_k), \qquad f(u) = u\big(1-u^2\big).$$

To construct the increments v_k we first make a heuristic analysis. If u were an exact solution, then the difference

$$\varepsilon = u - u_{k-1}$$

would solve the equation

$$\begin{aligned}\left(-\partial_t^2 + \partial_r^2 + \frac{1}{r}\partial_r\right)\varepsilon + \frac{2}{r^2}f'(u_{k-1})\varepsilon &= e_{k-1} - \frac{2}{r^2}\big(f(u_{k-1}+\varepsilon) - f'(u_{k-1})\varepsilon - f(u_{k-1})\big)\\ &= e_{k-1} + \frac{2}{r^2}\big(3\varepsilon^2 u_{k-1} + \varepsilon^3\big). \qquad (3.2)\end{aligned}$$

In a first approximation we linearize this equation around $\varepsilon = 0$ and substitute u_{k-1} by $Q(R)$. Then we obtain the linear approximate equation

$$\left(-\partial_t^2 + \partial_r^2 + \frac{1}{r}\partial_r + \frac{2}{r^2}\left(1 - 3Q^2\right)\right)\varepsilon \approx e_{k-1}. \tag{3.3}$$

For $r \ll t$ we expect the time derivative to play a lesser role so we neglect it and we are left with an elliptic equation with respect to the variable r,

$$\left(\partial_r^2 + \frac{1}{r}\partial_r + \frac{2}{r^2}\left(1 - 3Q^2\right)\right)\varepsilon \approx e_{k-1}, \quad r \ll t. \tag{3.4}$$

For $r \approx t$ we rewrite (3.3) in the form

$$\left(-\partial_t^2 + \partial_r^2 + \frac{1}{r}\partial_r - \frac{4}{r^2}\right)\varepsilon \approx e_{k-1}.$$

Here the time and spatial derivatives have the same strength. However, we can identify another principal variable, namely $a = r/t$ and think of ε as a function of (t, a). As it turns out, neglecting a "higher order" part of e_{k-1} which can be directly included in e_k, we are able to use scaling and the exact structure of the principal part of e_{k-1} to reduce the above equation to a Sturm–Liouville problem in a which becomes singular at $a = 1$.

The above heuristics lead us to a two step iterative construction of the v_k's. The two steps successively improve the error in the two regions $r \ll t$, respectively $r \approx t$. To be precise, we define v_k by

$$\left(\partial_r^2 + \frac{1}{r}\partial_r + \frac{2}{r^2}\left(1 - 3Q^2\right)\right)v_{2k+1} = e_{2k}^0, \tag{3.5}$$

respectively

$$\left(-\partial_t^2 + \partial_r^2 + \frac{1}{r}\partial_r - \frac{4}{r^2}\right)v_{2k} = e_{2k-1}^0 \tag{3.6}$$

both equations having zero Cauchy data[1] at $r = 0$. Here at each stage the error term e_k is split into a principal part and a higher order term (to be made precise below),

$$e_k = e_k^0 + e_k^1.$$

The successive errors are then computed as

$$e_{2k} = e_{2k-1}^1 + N_{2k}(v_{2k}), \qquad e_{2k+1} = e_{2k}^1 + \partial_t^2 v_{2k+1} + N_{2k+1}(v_{2k+1})$$

where

$$N_{2k+1}(v) = \frac{6}{r^2}\left(u_{2k}^2 - Q^2\right)v + \frac{2}{r^2}\left(3v^2 u_{2k} + v^3\right), \tag{3.7}$$

[1] The coefficients are singular at $r = 0$, therefore this has to be given a suitable interpretation.

respectively

$$N_{2k}(v) = \frac{6}{r^2}\left(u_{2k-1}^2 - 1\right)v + \frac{2}{r^2}\left(3v^2 u_{2k-1} + v^3\right). \tag{3.8}$$

To formalize this scheme we need to introduce suitable function spaces in the cone

$$\mathcal{C}_0 = \left\{(r,t)\colon\ 0 \leqslant r < t,\ 0 < t < t_0\right\}$$

for the successive corrections and errors. We first consider the a dependence. For the corrections v_k we use the following general concept.

Definition 3.2. Let $k \geqslant 0$. Then $\mathcal{Q}_k$ is the algebra of continuous functions $q(a, \alpha, \alpha_1)$

$$q : (0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

with the following properties:

(i) q is smooth in $a \in (0, 1)$, and meromorphic and even around $a = 0$. Further, the restriction to the diagonal

$$\tilde{q}(a, b) := q(a, b, b)$$

extends analytically to $a = 0$ and has an even expansion there.

(ii) q has the form

$$q(a, \alpha, \alpha_1) = \sum_{i+j \leqslant -k}^{j \leqslant 0,\, i \leqslant |j|/2} q_{ij}(a, \log\alpha, \log\alpha_1)\alpha^i \alpha_1^j$$

with q_{ij} polynomial in $\log\alpha, \log\alpha_1$. The sum only has finitely many terms.

(iii) Near $a = 1$ we have a representation of the form

$$q = q_0(a, \alpha) + \left(1 - a^2\right)^{\frac{1}{2}} q_1(a, \alpha, \alpha_1)$$

with coefficients q_0, q_1 analytic in a around $a = 1$.

The order of the pole at $a = 0$ as it appears in Definition 3.2, part (i), is controlled by some absolute constant depending only on k. The same comment applies to every pole at $a = 0$ appearing in this section and will be assumed tacitly throughout. For the errors e_k we introduce another functions class:

Definition 3.3. $\mathcal{Q}_k'$ is the space of continuous functions $q(a, \alpha, \alpha_1)$

$$q : (0, 1) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

with the following properties:

(i) q is smooth in $a \in (0,1)$, meromorphic and even around $a=0$. The restriction to the diagonal

$$\tilde{q}(a,b,b)$$

extends analytically to $a=0$.

(ii) q has a representation as in (ii) of the preceding definition.

(iii) Near $a=1$ we have a representation of the form

$$q = q_0(a,\alpha) + \left(1-a^2\right)^{\frac{1}{2}} q_1(a,\alpha,\alpha_1) + \left(1-a^2\right)^{-\frac{1}{2}} q_2(a,\alpha,\alpha_1)$$

with coefficients q_0, q_1, q_2 analytic with respect to a around $a=1$. Moreover, q_2 has the same representation as q in (ii), but with k replaced by $k+1$ and $j \leqslant -1$.

Next we define the class of functions of R:

Definition 3.4. $S^m(R^k(\log R)^\ell)$ is the class of analytic functions $v:[0,\infty) \to \mathbb{R}$ with the following properties:

(i) v vanishes of order m at $R=0$, and $R^{-m}v$ has an even expansion around $R=0$.

(ii) v has a convergent expansion near $R=\infty$,

$$v(R) = \sum_{0 \leqslant j \leqslant \ell+i} c_{ij}\, R^{k-2i} (\log R)^j.$$

Finally, we introduce the auxiliary variables

$$b := |\log t|, \qquad b_1 := |\log t| + \left|\log p(a)\right|$$

where p is a real even polynomial with the following properties:

$$p(1) = 0, \qquad p'(1) = -1, \qquad p(a) = 1 + O\left(a^M\right) \quad \text{as } a \to 0$$

where M is a very large number (depending on the number k of steps in our iteration), and p has no zeroes in $(0,1)$. We can now define the main function class for our construction.

Definition 3.5.

(a) $S^m(R^k(\log R)^\ell, \mathcal{Q}_n)$ is the class of analytic functions

$$v:[0,\infty) \times [0,1] \times \mathbb{R}^2 \to \mathbb{R}$$

(i) v is analytic as a function of R,

$$v:[0,\infty) \to \mathcal{Q}_n,$$

(ii) v vanishes of order m at $R=0$,

(iii) v has a convergent expansion at $R=\infty$,

$$v(R,\cdot,b,b_1)=\sum_{0\leqslant j\leqslant \ell+i} c_{ij}(\cdot,b,b_1)R^{k-2i}(\log R)^j$$

with coefficients $c_{ij}\in \mathcal{Q}_n$.

(b) $IS^m(R^k(\log R)^\ell,\mathcal{Q}_n)$ is the class of analytic functions w on the cone $\mathcal{C}_0$ which can be represented as

$$w(r,t)=v(R,a,b,b_1),\quad v\in S^m\big(R^k(\log R)^\ell,\mathcal{Q}_n\big).$$

We note that the representation of functions on the cone as in part (b) is in general not unique since R,a,b are dependent variables. Later we shall exploit this fact and switch from one representation to another as needed. We start our construction with some explicit computations which allow us to establish the regularity of the first few terms in the iteration, namely

$$v_1\in(t\lambda)^{-2}\frac{R^4}{4(1+R^2)^2}+(\lambda t)^{-2}\left(\frac{1}{|\log t|}IS^4\big(R^2\big)+\frac{1}{|\log t|^2}IS^4\big(R^2\big)\right), \tag{3.9}$$

$$t^2e_1\in(\lambda t)^{-2}\left(IS^4(1)+\frac{1}{|\log t|}IS^4\big(R^2\big)+\frac{1}{|\log t|^2}IS^4\big(R^2\big)\right), \tag{3.10}$$

$$v_2\in a^4IS(1,\mathcal{Q}_1). \tag{3.11}$$

After these few steps we reach the general pattern, and prove by induction that the successive corrections v_k and the corresponding error terms e_k can be chosen with the following properties:

$$v_{2k-1}\in IS^4\big(R^2(\log R)^{k-1},\mathcal{Q}_{2\beta k}\big), \tag{3.12}$$

$$t^2e_{2k-1}\in IS^2\big(R^2(\log R)^{k-1},\mathcal{Q}'_{2\beta k}\big), \tag{3.13}$$

$$v_{2k}\in a^4IS\big((\log R)^{k-1},\mathcal{Q}_{2\beta(k-1)}\big), \tag{3.14}$$

$$t^2e_{2k}\in a^2IS\big((\log R)^{k-1},\mathcal{Q}'_{2\beta k}\big)+IS^4\big((\log R)^{k-1},\mathcal{Q}_{2\beta k}\big). \tag{3.15}$$

The properties (3.9)–(3.15) suffice in order to reach the conclusion of the theorem. We note that is easy to verify that all the above classes of functions are left invariant by the scaling operator S.

The proof of (3.9)–(3.15) roughly follows that in [11,12]. There is, however, an important difference near the light cone: for the critical Wave Maps problem as well as the critical focussing semilinear equation, the singularity at the boundary of the light cone is well modeled by the expression $(1-a)^{\frac{1}{2}+\nu}$, which comes from the choice of blow up rate $t^{-1-\nu}$. For Yang–Mills, due to the much faster blow up speed, we need to essentially use the much more singular expression

$$\frac{(1-a)^{\frac{1}{2}}}{|\log t|+|\log p(a)|} \tag{3.16}$$

where $p(a)$ is a polynomial so that $p(1)=0$. This renders the algebra significantly more delicate. We remark that (3.16) appears canonically in this section. On one hand, $(1-a)^{\frac{1}{2}}$ is part of a fundamental system of that ODE which (3.6) reduces to in the self-similar variable $a=\frac{r}{t}$. This is exactly what one would obtain by neglecting all but the selfsimilar components of the wave operator. However, unlike in [11,12], here we encounter a nontrivial non-selfsimilar effect which forces the exact denominator in (3.16). In particular this saves the day by insuring that (3.16) belongs to $H^1(0,1)$ which of course is a minimal requirement here.

To commence the construction of the v_k, we recall that

$$Q(R) = \frac{1-R^2}{1+R^2}, \qquad \Phi(R) = \frac{R^2}{(1+R^2)^2}$$

where Φ is the zero eigenfunction, $L\Phi = 0$ with

$$L = \partial_R^2 + \frac{1}{R}\partial_R + \frac{2}{R^2}\bigl(1 - 3Q^2\bigr).$$

By (3.1) we have

$$t^2 e_0 = -t^2\partial_t^2 Q(R) = \left(1 + \frac{\beta}{|\log t|}\right)^2 R\Phi'(R) + \left(1 + \frac{\beta}{|\log t|} - \frac{\beta}{|\log t|^2}\right)\Phi(R).$$

Step 1. *Begin with e_0 as above and choose v_1 so that* (3.12) *for $k=1$ holds. Further the error e_1 thereby generated is of the form* (3.13) *for $k=1$.*

Here, we simply put $e_0^0 := e_0$. Reformulate the equation for v_1 as follows:

$$(t\lambda)^2 \tilde{L}\sqrt{R}v_1 = \sqrt{R}t^2 e_0, \quad \tilde{L} = \partial_R^2 - \frac{15}{4R^2} + \frac{24}{(1+R^2)^2}.$$

Using the above calculation of e_0, it is then straightforward to write down an absolutely convergent Taylor expansion of v_1 around $R=0$. Since $t^2 e_0$ vanishes of second order at 0, it follows that v_1 vanishes of order four at 0.

Now we turn to the expansion around $R=\infty$. The leading term in $t^2 e_0$ is $R\Phi'(R) + \Phi(R)$. A key fact is that this satisfies the orthogonality condition

$$\bigl\langle R\Phi'(R) + \Phi(R), \Phi\bigr\rangle_{\mathbb{R}^2} = 0.$$

It is partly this orthogonality condition which motivates our choice of $\lambda(t)$. As a consequence, the solution to $Lv_{10} = R\Phi'(R) + \Phi(R)$ does not grow at ∞, precisely it equals

$$v_{10} = \frac{1}{4}(t\lambda)^{-2}\frac{R^4}{(1+R^2)^2}.$$

For the remaining terms we do not have such a precise representation since we lack the orthogonality condition. We use this fundamental system of solutions for $\tilde{L}$:

$$\phi_0(R) = \frac{R^{\frac{5}{2}}}{(1+R^2)^2}, \qquad \theta_0(R) = \frac{-1 - 8R^2 + 24R^4\log R + 8R^6 + R^8}{4R^{\frac{3}{2}}(1+R^2)^2}.$$

Their Wronskian is $W(\phi_0, \theta_0) = 1$. Then $\Phi(R) = R^{-\frac{1}{2}}\phi_0(R)$ and define $\Theta(R) := R^{-\frac{1}{2}}\theta_0(R)$ so that $L\Phi = 0$, $L\Theta = 0$, respectively.[2] One thus obtains an integral representation for v_1 using the variation of parameters formula, which gives

$$\begin{aligned}(t\lambda)^2 v_1(R) &= \frac{\theta_0(R)}{\sqrt{R}} \int_0^R \phi_0(R')\sqrt{R'}t^2 e_0(R')\,dR' - \frac{\phi_0(R)}{\sqrt{R}} \int_1^R \theta_0(R')\sqrt{R'}t^2 e_0(R')\,dR' \\ &= \Theta(R) \int_0^R \Phi(R')t^2 e_0(R')R'\,dR' - \Phi(R) \int_1^R \Theta(R')t^2 e_0(R')R'\,dR'.\end{aligned}$$

In the end we obtain the representation

$$v_1 = v_{10} + v_{11}, \qquad v_{11} \in (\lambda t)^{-2}\left(\frac{1}{|\log t|} IS^4\big(R^2\big) + \frac{1}{|\log t|^2} IS^4\big(R^2\big)\right) \tag{3.17}$$

which implies (3.9).

Next, we determine the error, which is given by

$$t^2 e_1 = t^2 \partial_t^2 v_1 - \frac{3(\lambda t)^2}{R^2}\big(3v_1^2 Q + v_1^3\big).$$

After some computations we obtain the relation (3.10), namely

$$t^2 e_1 \in (\lambda t)^{-2}\left(IS^4(1) + \frac{1}{|\log t|} IS^4\big(R^2\big) + \frac{1}{|\log t|^2} IS^4\big(R^2\big)\right). \tag{3.18}$$

Step 2. Recall that v_2 is determined by (3.6), which requires specifying e_1^0. This will be done iteratively, which means that

$$e_1^0 = \sum_{j=0}^{J} e_1^{0j} \tag{3.19}$$

where $J = J(\beta)$ grows with β and e_1^{0j} is specified recursively. To being with, we extract the leading order (in terms of growth in R) from e_1 and set

$$t^2 e_1^{00} := c_1(\lambda t)^{-2}\frac{1}{|\log t|}R^2 + c_2(\lambda t)^{-2}\frac{1}{|\log t|^2}R^2 = c_1\frac{a^2}{b} + c_2\frac{a^2}{b^2}$$

with suitable constants c_1, c_2. Note that then

$$e_1^{10} := e_1 - e_1^{00} \in IS^2(1, \mathcal{Q}_{2\beta})$$

[2] Note the appearance of $\Phi(R)\log R$ as part of Θ.

which is admissible for e_2, see (3.15). Replacing Q by 1 we now seek to solve the linear differential equation

$$t^2\left(-\partial_t^2 + \partial_r^2 + \frac{1}{r}\partial_r - \frac{4}{r^2}\right)v_2 = t^2 e_1^{00}. \tag{3.20}$$

In the a, b coordinates the above equation is rewritten as

$$L_{ab} v_2(a,b) = c_1 \frac{a^2}{b} + c_2 \frac{a^2}{b^2}$$

where

$$L_{ab} = -(\partial_b + a\partial_a)^2 - (\partial_b + a\partial_a) + \partial_a^2 + \frac{1}{a}\partial_a - \frac{4}{a^2}.$$

Set also the b independent part

$$L_a = (1-a^2)\partial_a^2 + \left(\frac{1}{a} - 2a\right)\partial_a - \frac{4}{a^2}.$$

For technical reasons, we will only obtain an approximate solution v_2 of (3.20). We then face a dichotomy: either the error $L_{ab}\, v_2 - e_1^{00}$ is acceptable for e_2 or not; in the latter case, we repeat the procedure by including the unacceptable error in e_1^0 and solving for a correction to v_2. This process (which also needs to take the nonlinear component of e_2 into account, see (3.8)) then leads to the aforementioned iterative construction of e_1^0 and v_2.

We begin by constructing an approximate solution to $L_{ab} w_2 = \frac{a^2}{b}$. The approximate solution in the following lemma is called w_2 rather than v_2 since the latter will be the sum of various expressions, the first being w_2.

Lemma 3.6. *Let $e(a)$ be even analytic and quadratic at $a = 0$. There is an approximate solution w_2 for*

$$L_{ab} w_2 = b^{-1} e(a)$$

which is of the form

$$w_2(a,b) = b^{-1} W_2^0(a) + b_1^{-1}(1-a)^{\frac{1}{2}} W_2^1(a) \tag{3.21}$$

where W_2^0, W_2^1 are even analytic in $a \in (0,1]$ with an a^{-2} leading term at 0 so that w_2 vanishes to fourth order at $a = 0$. The error has the form

$$\begin{aligned} f_2^0 &:= L_{ab} w_2 - b^{-1} e(a) \\ &= E_2^0(a, b^{-1}) + (1-a)^{\frac{1}{2}} E_2^1\big(a, b_1^{-1}\big) \end{aligned} \tag{3.22}$$

where E_2^0, E_2^1 are analytic in $a \in (0,1]$, linear in b^{-2}, b_1^{-2}, b^{-3}, b_1^{-3}, and vanish quadratically at $a = 0$.

Proof. We begin with the ansatz

$$w_2 = \frac{W_2^0(a)}{b} + \frac{(1-a)^{\frac{1}{2}} W_2^1(a)}{b_1}$$

where

$$L_a W_2^0(a) = e(a), \qquad L_a\big((1-a)^{\frac{1}{2}} W_2^1(a)\big) = 0.$$

The solvability of these equations will be discussed later in the proof. Then

$$L_{ab} w_2 = \frac{L_a W_2^0(a)}{b} + \frac{L_a((1-a)^{\frac{1}{2}} W_2^1(a))}{b_1} + f_2^0 = \frac{e(a)}{b} + f_2^0$$

where

$$\begin{aligned}
f_2^0 &= \big(-\partial_b^2 - 2\partial_b a\partial_a - \partial_b\big)\frac{W_2^0(a)}{b} - (1-a)^{\frac{1}{2}} W_2^1(a)\big(\partial_b^2 + \partial_b\big)\left[\frac{1}{b_1}\right] \\
&\quad + (1-a)^{\frac{1}{2}} W_2^1(a)\big(\big(a^{-1} - 2a + 1\big)\partial_a - (1-a)^2\partial_a^2\big)\left[\frac{1}{b_1}\right] \\
&\quad + 2\big(1-a^2\big)\partial_a\big((1-a)^{\frac{1}{2}} W_2^1(a)\big)\partial_a\left[\frac{1}{b_1}\right] - 2a(1-a)^{\frac{1}{2}}\partial_a W_2^1(a)\partial_b\left[\frac{1}{b_1}\right] \\
&\quad - 2a W_2^1(a)\partial_a\partial_b\left[\frac{(1-a)^{\frac{1}{2}}}{b_1}\right] + (1-a)^{\frac{1}{2}} W_2^1(a)\big(-\partial_a + \big((1-a)^2 + \big(1-a^2\big)\big)\partial_a^2\big)\left[\frac{1}{b_1}\right].
\end{aligned}$$

The final term here is the same as

$$\begin{aligned}
&(1-a)^{\frac{1}{2}} W_2^1(a)\big(-\partial_a + 2(1-a)\partial_a^2\big)\left[\frac{1}{b_1}\right] \\
&\quad = W_2^1(a)\big(-\partial_a + 2(1-a)\partial_a^2\big)\left[\frac{(1-a)^{\frac{1}{2}}}{b_1}\right] + 2(1-a)^{\frac{1}{2}} W_2^1(a)\partial_a\left[\frac{1}{b_1}\right]
\end{aligned}$$

which implies that the error equals

$$\begin{aligned}
f_2^0 &= \big(-\partial_b^2 - 2\partial_b a\partial_a - \partial_b\big)\frac{W_2^0(a)}{b} - (1-a)^{\frac{1}{2}} W_2^1(a)\big(\partial_b^2 + \partial_b\big)\left[\frac{1}{b_1}\right] \\
&\quad + (1-a)^{\frac{1}{2}} W_2^1(a)\big(\big(a^{-1} - 2a + 1\big)\partial_a - (1-a)^2\partial_a^2\big)\left[\frac{1}{b_1}\right] \\
&\quad + (1-a)^{\frac{3}{2}}\big(W_2^1(a) + 2(1+a)\partial_a W_2^1(a)\big)\partial_a\left[\frac{1}{b_1}\right] \qquad (3.23) \\
&\quad - 2a(1-a)^{\frac{1}{2}}\partial_a W_2^1(a)\partial_b\left[\frac{1}{b_1}\right]
\end{aligned}$$

$$+ W_2^1(a)\big(-2a\partial_b\partial_a + 2(1-a)\partial_a^2 - \partial_a\big)\left[\frac{(1-a)^{\frac{1}{2}}}{b_1}\right]. \tag{3.24}$$

In the first term we gain at least one power of b^{-1}. In the second and fifth terms we gain at least one power of b_1^{-1}. Since

$$(1-a)\partial_a b_1 = -\frac{(1-a)p'(a)}{p(a)}$$

which is analytic in [0, 1] it follows that in the third and fourth terms we gain at least one power of b_1^{-1} without losing any power of $(1-a)$.

So far we have considered the negligible terms. The key expression is the one in the final term, which determines the choice of our ansatz. Here there is a nontrivial cancellation which yields an additional $1-a$ factor. To begin with, recall that

$$\big(2(1-a)\partial_a^2 - \partial_a\big)(1-a)^{\frac{1}{2}} = 0.$$

This implies that in (3.24) at least one derivative has to fall on b_1 leading to a gain of at least one power of b_1. However, we need to check that there is no loss in terms of powers of $(1-a)$. This can be seen via the factorization (we first consider $\partial_a\partial_b$ since the difference from $a\partial_a\partial_b$ gains a factor of $1-a$)

$$\begin{aligned}
&\big(-2\partial_b\partial_a + 2(1-a)\partial_a^2 - \partial_a\big)(1-a)^{\frac{1}{2}} g(a,b) \\
&\quad = \big(2(1-a)^{\frac{1}{2}}\partial_a - (1-a)^{-\frac{1}{2}}\big)\big(-\partial_b + (1-a)\partial_a\big) g(a,b)
\end{aligned} \tag{3.25}$$

provided $g(a,b)$ is smooth. In particular, setting $g(a,b) = \frac{1}{b_1}$,

$$\begin{aligned}
&\big(-2\partial_b\partial_a + 2(1-a)\partial_a^2 - \partial_a\big)\frac{(1-a)^{\frac{1}{2}}}{b_1} \\
&\quad = \big(2(1-a)^{\frac{1}{2}}\partial_a - (1-a)^{-\frac{1}{2}}\big)\big(-\partial_b + (1-a)\partial_a\big)\left[\frac{1}{b_1}\right] \\
&\quad = \big(2(1-a)^{\frac{1}{2}}\partial_a - (1-a)^{-\frac{1}{2}}\big) b_1^{-2}\big(\partial_b - (1-a)\partial_a\big) b_1 \\
&\quad = \left(2\frac{(1-a)^{\frac{1}{2}}}{b_1^2}\partial_a - 4\frac{(1-a)^{\frac{1}{2}}}{b_1^3}\partial_a b_1 - \frac{(1-a)^{-\frac{1}{2}}}{b_1^2}\right)\big(\partial_b - (1-a)\partial_a\big) b_1.
\end{aligned} \tag{3.26}$$

Given our choice of b_1,

$$\big(-\partial_b + (1-a)\partial_a\big) b_1 = -1 - \frac{p'(a)}{p(a)}(1-a) = O(1-a).$$

Thus, the $(1-a)$-gain in the second factor in (3.26) cancels the $(1-a)$-loss that we incur in the first factor. At the same time we get at least a b_1^{-2} factor. In conclusion,

$$\big(-2a\partial_b\partial_a + 2(1-a)\partial_a^2 - \partial_a\big)\frac{(1-a)^{\frac{1}{2}}}{b_1} = O\left(\frac{(1-a)^{\frac{1}{2}}}{b_1^2}\right) \tag{3.27}$$

where the $O(\cdot)$-term here depends linearly on b_1^{-2} and b_1^{-3}. This establishes the desired estimate on the error f_2^0.

We now consider the principal part, for which we need to solve

$$L_a W_2^0 = e(a), \qquad L_a\big((1-a)^{\frac{1}{2}} W_2^1(a)\big) = 0. \tag{3.28}$$

In order to analyze these equations, we first discuss fundamental systems of L_a and their respective behaviors at the regular singular points $a=0$ and $a=1$ of L_a (we can ignore the regular singular point $a=-1$ of L_a). From

$$L_a\big(a^k\big) = \big(k^2-4\big)a^{k-2} - k(k+1)a^k$$

we conclude that $L_a[a^{\pm 2}(1+a^2\phi_\pm(a))] = 0$ where $\phi_\pm$ are even analytic functions around $a=0$. Moreover, a particular solution to $L_a(f) = a^2$ is given by $f(a) = -\frac{a^2}{6}$. Similarly, for any $e(a)$ as in the statement of the lemma there is a particular solution $f(a)$ to $L_a f = e$ with f even analytic around $a=0$ and vanishing quadratically at $a=0$. Note that f is not unique. However, adding a suitable multiple of the a^2-homogeneous solution we can achieve that $f(a)$ vanishes to fourth order at $a=0$ (i.e. $f(a) = O(a^4)$) and is unique.

To analyze a fundamental system around $a=1$ we write

$$\begin{aligned} L_a &= 2(1-a)^{\frac{1}{2}}\partial_a\big((1-a)^{\frac{1}{2}}\partial_a\big) - (1-a)^2\partial_a^2 + a^{-1}(1+2a)(1-a)\partial_a - \frac{4}{a^2} \\ &=: L_{a,0} + L_{a,1} \end{aligned}$$

where $L_{a,0} := 2(1-a)^{\frac{1}{2}}\partial_a((1-a)^{\frac{1}{2}}\partial_a)$. Now

$$\begin{aligned} L_{a,0}(1-a)^k &= k(2k-1)(1-a)^{k-1}, \\ L_a(1-a)^k &= k(2k-1)(1-a)^{k-1} + O\big((1-a)^k\big) \end{aligned}$$

with an analytic $O(\cdot)$-term. This implies that $L_a\psi_0 = L_a\psi_1 = 0$ with

$$\psi_0(a) = 1 + (1-a)\tilde{\psi}_0(a), \qquad \psi_1(a) = (1-a)^{\frac{1}{2}}\big(1 + (1-a)\tilde{\psi}_1(a)\big) \tag{3.29}$$

where $\tilde{\psi}_0$, $\tilde{\psi}_1$ are analytic around $a=1$. In particular, we can solve for W_2^1 in (3.28) and W_2^1 is unique up to a constant factor. For future reference we remark that

$$L_a = \rho_1\partial_a(\rho_2\partial_a) - \frac{4}{a^2}, \quad \rho_1(a) = \frac{1}{a}\sqrt{1-a^2},\ \rho_2(a) = a\sqrt{1-a^2}.$$

To solve (3.28), we first solve for $W := W_2^0 + (1-a)^{\frac{1}{2}} W_2^1$ and then extract W_2^0 and W_2^1 from it. The logic here is as follows: At $a = 0$ we want w_2 to vanish to fourth order. This implies that W must also vanish to the same order since

$$b_1 - b = \left|\log p(a)\right| = -\log\bigl(1 - O\bigl(a^M\bigr)\bigr) = O\bigl(a^M\bigr)$$

with M large. Therefore, as discussed above, W is uniquely determined as a solution to

$$L_a W(a) = a^2, \quad -\varepsilon < a < 1,$$

where $\varepsilon > 0$ is some small constant. By variation of parameters there exist unique constants c_0, c_1, c_2 with the property that

$$W(a) = c_0\psi_0(a) + c_1\psi_1(a) + c_2 \int_a^1 \bigl[\psi_0(a)\psi_1(u) - \psi_1(a)\psi_0(u)\bigr]\bigl(\rho_1(u)\bigr)^{-1} u^2\, du.$$

By inspection, the integral on the right-hand side is smooth around $a = 1$. This shows that we need to set

$$W_2^1(a) := c_1\psi_1(a),$$
$$W_2^0(a) := c_0\psi_0(a) + c_2 \int_a^1 \bigl[\psi_0(a)\psi_1(u) - \psi_1(a)\psi_0(u)\bigr]\bigl(\rho_1(u)\bigr)^{-1} u^2\, du.$$

Observe that at $a = 0$ we have no guarantee that W_2^0, W_2^1 are smooth; in fact, they may exhibit a^{-2}-type behavior. $\square$

We remind the reader that b_1 in (3.21) cannot be replaced with b since we require that $w_2 \in H^1(0,1)$ relative to the a variable. The proof also shows that one cannot dispense with the $(1-a)^{\frac{1}{2}}$ part of w_2 since it is part of the fundamental system of L_a. Another important feature of the previous proof is the cancellation in (3.25). For our purposes, $g(a,b) = h(b_1)$ whence (3.25) becomes

$$\begin{aligned}
&\bigl(-2\partial_b\partial_a + 2(1-a)\partial_a^2 - \partial_a\bigr)(1-a)^{\frac{1}{2}} h(b_1) \\
&\quad = \bigl(2(1-a)^{\frac{1}{2}}\partial_a - (1-a)^{-\frac{1}{2}}\bigr)h'(b_1)\bigl(-\partial_b + (1-a)\partial_a\bigr)b_1 \\
&\quad = \bigl(2(1-a)^{\frac{1}{2}} h''(b_1)\partial_a b_1 + 2(1-a)^{\frac{1}{2}} h'(b_1)\partial_a - h'(b_1)(1-a)^{-\frac{1}{2}}\bigr)O(1-a) \\
&\quad = O\bigl((1-a)^{\frac{1}{2}} h''(b_1)\bigr) + O\bigl((1-a)^{\frac{1}{2}} h'(b_1)\bigr).
\end{aligned} \tag{3.30}$$

In view of (3.30), the proof of Lemma 3.6 generalizes to right-hand sides such as $\frac{e(a)}{b^k}$ for any $k \geqslant 1$.

If we were to now set $v_2 := w_2$ (from the previous lemma), then the error f_2^0 from (3.22), as well as the remaining $c_2\frac{a^2}{b^2}$ piece from e_1^{00}, would have to be included in e_2. However, if

$\beta > 1$ this is inadmissible since the error e_2 needs to decay at least like $(t\lambda(t))^{-2} = b^{-2\beta}$. The importance of $(t\lambda)^{-2}$ lies with scaling; indeed, the elliptic equation (3.5) scales like R^2 which equals $(t\lambda)^2$ at its largest.

These are not the only obstacles we face here: the nonlinear part of e_2 (again if $v_2 = w_2$) is

$$\frac{6}{r^2}\big(u_1^2 - 1\big)w_2 + \frac{2}{r^2}\big(3w_2^2 u_1 + w_2^3\big), \tag{3.31}$$

where $u_1 = Q + v_1$, see (3.8). One easily checks that the preceding expression times t^2 lies in

$$(\lambda t)^{-2} IS^4(1, \mathcal{Q}_1) + (\lambda t)^{-2} IS^4\big(R^2, \mathcal{Q}_2\big).$$

The term $(\lambda t)^{-2} IS^4(1, \mathcal{Q}_1)$ can be incorporated into $t^2 e_2$; however, the term

$$(\lambda t)^{-2} IS^4\big(R^2, \mathcal{Q}_2\big)$$

is not acceptable for e_2 due to the R^2 growth.

We deal with these obstacles by including all unacceptable errors e (with regard to e_2) in e_1^0 and solving $L_{ab}\, w = e$. For example, using the notation of Lemma 3.6 the second term in (3.31) contributes

$$e = a^{-2} \frac{(1-a)^{\frac{1}{2}} W_2^0(a) W_2^1(a)}{bb_1}$$

where we replaced u_1 with 1. The corresponding ansatz for w would then necessarily contain the term

$$w = (bb_1)^{-1}(1-a)^{\frac{1}{2}} W(a).$$

If $\partial_a \partial_b$ (which is part of L_{ab}) hits this term, then we obtain (amongst others) the error term

$$(1-a)^{-\frac{1}{2}} b^{-1} b_1^{-2}.$$

Iterating once more with this error on the right-hand side produces the expression

$$(1-a)^{\frac{1}{2}} b_1^{-2} \log b.$$

In order to remove possible singularities at $a = 0$ (as in the previous proof) one needs as many powers of $\log b_1$ as of $\log b$. These observations should serve to motivate the following result which will finally allow us to carry out the full iteration for v_2 (as well as for v_{2k} in Step 4 below). We begin with a definition.

Definition 3.7. Let $2k \geqslant m \geqslant k \geqslant 1$. By $\mathcal{F}_{k,m}$ we mean the function class

$$\mathcal{F}_{k,m} := \Biggl\{ f_k \,\Bigg|\, f_k = b^{-k} e_0(a, \log b) + (1-a)^{\frac{1}{2}} \sum_{j=1}^{m} e_j^0(a, \log b, \log b_1) b^{j-k} b_1^{-j}$$
$$+ (1-a)^{-\frac{1}{2}} \sum_{j=1}^{m} e_j^1(a, \log b, \log b_1) b^{j-k-1} b_1^{-j} \Biggr\}$$

where for each j the functions e_0, $e_j^0(a)$, $e_j^1(a)$ are smooth in $a \in (0,1)$, analytic around $a = 1$, meromorphic and even around $a = 0$. Moreover, these functions are polynomials in the variables $\log b$, and $\log b_1$, respectively. Further, $f_k = O(a^2)$ as $a \to 0$.

Recall that the order of the pole at $a = 0$ is controlled by a constant depending only on k. In what follows, we will tacitly assume that M in the definition of b_1 is sufficiently large depending on k (in fact, the order of the pole at $a = 0$ in the previous definition). Since we are only going to consider finitely many k, this is not an issue. Since $\log b_1 - \log b = O(a^M)$ we see that $f_k(a) = O(a^2)$ is therefore the same as

$$e_0(a, \log b) + \sum_{j=1}^{m} (1-a)^{\frac{1}{2}} e_j^0(a, \log b, \log b) + (1-a)^{-\frac{1}{2}} \sum_{j=1}^{m} b^{-1} e_j^1(a, \log b, \log b) = O\bigl(a^2\bigr).$$

The left-hand side is a polynomial in $\log b$, b^{-1}, so this amounts to the corresponding condition for each of its coefficients. Now for the main iterative lemma.

Lemma 3.8. *The equation*

$$L_{ab}\, v = f_k \in \mathcal{F}_{k,m} \tag{3.32}$$

admits an approximate solution

$$v(a,b) = b^{-k} V_0(a, \log b) + (1-a)^{\frac{1}{2}} \sum_{j=1}^{m} V_j(a, \log b, \log b_1) b^{j-k} b_1^{-j}$$

where V_0, V_j are smooth in $a \in (0,1)$, analytic around $a = 1$, meromorphic around $a = 0$, and polynomial in the variables $\log b$, $\log b_1$. Moreover, v vanishes to fourth order at $a = 0$ and

$$L_{ab} v - f_k \in \mathcal{F}_{k+1,m} + \mathcal{F}_{k+2,m}.$$

Proof. Let $\ell(0)$ be the order of the polynomials appearing in the definition of f_k relative to $\log b$, and $\ell(j)$ the order relative to $\log b_1$ with $1 \leqslant j \leqslant m$. We first re-write the source term: choose a smooth partition of unity $\phi_{1,2}(a)$, subordinate to the cover $(0,1) = (0, 2\varepsilon) \cup (\varepsilon, 1)$ for some small $\varepsilon > 0$. Then write

$$\phi_1(a) \Biggl[b^{-k} e_0(a, \log b) + (1-a)^{\frac{1}{2}} \sum_{j=1}^{m} e_j^0(a, \log b, \log b_1) b^{j-k} b_1^{-j}$$

$$+ (1-a)^{-\frac{1}{2}} \sum_{j=1}^{m} e_j^1(a, \log b, \log b_1) b^{j-k-1} b_1^{-j} \Bigg]$$

$$= \phi_1(a) \Bigg[b^{-k} e_0(a, \log b) + (1-a)^{\frac{1}{2}} \sum_{j=1}^{m} e_j^0(a, \log b, \log b) b^{-k} \tag{3.33}$$

$$+ (1-a)^{-\frac{1}{2}} \sum_{j=1}^{m} e_j^1(a, \log b, \log b) b^{-k-1} \Bigg] \tag{3.34}$$

$$+ (\log b - \log b_1)\phi_1(a) \tilde{f}_k + (b - b_1)\phi_1(a) \tilde{g}_{k+1} \tag{3.35}$$

where $\tilde{f}_k$, $\tilde{g}_k$ have the same properties as f_k. Note that in the expression in brackets in (3.33) and (3.34), all singular powers cancel. For (3.35), expand

$$\phi_1(a)[\log b_1 - \log b] = \phi_1(a) \log\left(1 - \frac{\log|p(a)|}{b}\right) = -\phi_1(a) \sum_{j=1}^{N} \left[\frac{(\frac{\log|p(a)|}{b})^j}{j} \right] + \text{error}.$$

Here we may achieve arbitrarily fast decay in time for the error term upon choosing N large enough, and hence we can discard its contribution. However, now all the terms in

$$\phi_1(a) \big(\log|p(a)|\big)^j \tilde{f}_k, \qquad \phi_1(a) \big(\log|p(a)|\big)^j \tilde{g}_{k+1}, \quad j \geqslant 1,$$

are smooth up to $a = 0$, and so are all terms in

$$\phi_2(a) f_k = \phi_2(a) \Bigg[b^{-k} e_0(a, \log b) + (1-a)^{\frac{1}{2}} \sum_{j=1}^{m} e_j^0(a, \log b, \log b_1) b^{j-k} b_1^{-j}$$
$$+ (1-a)^{-\frac{1}{2}} \sum_{j=1}^{m} e_j^1(a, \log b, \log b_1) b^{j-k-1} b_1^{-j} \Bigg].$$

These considerations show that we may as well assume that e_0, e_j^0, e_j^1 are each analytic at $a = 0$ as well as of the form $O(a^2)$. With v as in the statement of the lemma, we compute

$$L_{ab} v = b^{-k} L_a V_0(a, \log b) + \sum_{j=1}^{m} b^{j-k} b_1^{-j} L_a\big((1-a)^{\frac{1}{2}} V_j(a, \log b, \log b_1)\big)$$
$$+ \sum_{j=1}^{m} a \partial_b \big(b^{j-k} b_1^{-j} (1-a)^{-\frac{1}{2}} V_j(a, \log b, \log b_1)\big) + \text{error}$$

where b_1 is treated as a parameter, i.e., no derivatives fall on it. Here the last term comes from $\partial_a \partial_b$ in L_{ab} with the ∂_a applied to $(1-a)^{\frac{1}{2}}$ and ∂_b applied to b or $\log b$. Assuming that V_j are smooth and that v vanishes of order four at $a = 0$ one sees that the error has the desired form

$$\text{error} \in a^2 \mathcal{Q}_{k+1}.$$

This is done using the same type of calculations leading to (3.23) and the following properties, cf. (3.30),

$$\big(-2\partial_b\partial_a + 2(1-a)\partial_a^2 - \partial_a\big)\frac{(1-a)^{\frac{1}{2}}}{b_1^k} = O\bigg(\frac{(1-a)^{\frac{1}{2}}}{b_1^{k+1}}\bigg),$$
$$\big(-2\partial_a\partial_b + 2(1-a)\partial_a^2 - \partial_a\big)\big[(1-a)^{\frac{1}{2}}(\log b_1)^k\big] = O\bigg(\frac{(1-a)^{\frac{1}{2}}(\log b_1)^{k-1}}{b_1^2}\bigg).$$

We also observe that in the second sum in $L_{ab}v$ only $V_j(1,\log b,\log b_1)$ is important. The rest can be also assigned to the error. Thus matching the $(1-a)^{\frac{1}{2}}$ like terms we are left with the equations

$$L_a V_0(a,\log b) = e_0(a,\log b),$$
$$L_a\big(V_j(a,\log b,\log b_1)(1-a)^{\frac{1}{2}}\big) = e_j^0(a,\log b,\log b_1)(1-a)^{\frac{1}{2}}.$$

Matching the $(1-a)^{-\frac{1}{2}}$ at $a=1$ we get the boundary conditions (recall that b_1 here is treated as a parameter)

$$\partial_b\big(b^{j-k}V_j(1,\log b,\log b_1)\big) = b^{j-k-1}e_j^1(1,\log b,\log b_1), \quad j=1,\dots,m. \tag{3.36}$$

More explicitly, (3.36) means the following. Separating into monomials in $\log b_1$ we seek s' and $\{c_\ell\}_{\ell=0}^{s'}$ so that

$$\partial_b\Bigg(b^{j-k}\sum_{\ell=0}^{s'} c_\ell \log^\ell b\Bigg) = b^{j-k-1}\sum_{\ell=0}^{s} c_\ell^0 \log^\ell b$$

for given s and $\{c_\ell^0\}_{\ell=0}^s$. If $j>k$ then we set $s' := s$ and

$$(j-k)c_\ell + (\ell+1)c_{\ell+1} = c_\ell^0, \quad 0 \leqslant \ell < s,$$
$$c_s = \frac{c_s^0}{j-k}$$

whereas in case $j=k$ we set $s' := s+1$ and $c_\ell = \frac{c_{\ell-1}^0}{\ell}$ for all $1 \leqslant \ell \leqslant s'$ (in particular, we generate extra powers of $\log b$ in this case and c_0 is not determined). Write

$$e_0(a,\log b) = \sum_{j=0}^{\ell(0)} P_j(a)\log^j b$$

with $P_j(a)$ is smooth on [0, 1], analytic close to $a=0$, and $P_j(a) = O(a^2)$. Then we solve the problems

$$L_a V_{0,j} = P_j, \quad j=0,\dots,\ell(0),$$

where we select a solution which is smooth at $a=1$. Using the notations of (3.29) and variation of parameters,

$$V_{0,j}(a) = c\psi_0(a) + c_0 \int_a^1 \bigl[\psi_0(a)\psi_1(u) - \psi_0(u)\psi_1(a)\bigr]\bigl(\rho_1(u)\bigr)^{-1} P_j(u)\,du$$

where c_0 is an absolute, and c an arbitrary, constant. Note that around $a=0$,

$$V_{0,j}(a) = O\bigl(a^2\bigr) + c_{0,j}\varphi_0(a)$$

where $L_a\varphi_0 = 0$ and $\varphi_0(a) = a^{-2}(1+O(a^2))$ with analytic $O(a^2)$ (as can be seen from a power series ansatz). Then define

$$V_0(a, \log b) := \sum_{j=0}^{\ell(0)} V_{0,j}(a) \log^j b.$$

Even though this expression will in general be singular at $a=0$, the singular part is of the form

$$\varphi_0(a) \sum_{j=0}^{\ell(0)} c_{0,j} \log^j b.$$

Similarly, we write

$$e_j^0(a, \log b, \log b_1) = \sum_{k=0}^{\ell(j)} \sum_{\ell+n=k} q_{j,\ell,n}(a) \log^\ell b \log^n b_1$$

where $q_{j,\ell,n}$ are smooth, analytic around $a=0$ and vanishing to second order at $a=0$, and solve the problems

$$L_a\bigl[(1-a)^{\frac{1}{2}} V_{j,\ell,n}(a)\bigr] = (1-a)^{\frac{1}{2}} q_{j,\ell,n}(a)$$

by variation of parameters, i.e.,

$$\begin{aligned}
&(1-a)^{\frac{1}{2}} V_{j,\ell,n}(a) \\
&\quad = c_{j,\ell,n}\,\psi_1(a) + c_0 \int_a^1 \bigl[\psi_0(a)\psi_1(u) - \psi_0(u)\psi_1(a)\bigr]\bigl(\rho_1(u)\bigr)^{-1}(1-u)^{\frac{1}{2}} q_{j,\ell,n}(u)\,du
\end{aligned}$$

where $c_{j,\ell,n}$ is arbitrary. Note that $V_{j,\ell,n}(a)$ is smooth around $a=1$. As for the behavior around $a=0$, one has

$$(1-a)^{\frac{1}{2}} V_{j,\ell,n}(a) = O(a^2) + c\varphi_0(a)$$

as before. Moreover, since

$$(1-a)^{-\frac{1}{2}}\psi_1(a) = 1 + O(1-a)$$

we conclude that $V_{j,\ell,n}(1)$ can be assigned arbitrary values. This is crucial with regard to the boundary condition (3.36). More precisely, setting

$$V_j(a, \log b, \log b_1) := \sum_{k=0}^{\ell(j)} \sum_{\ell+n=k} V_{j,\ell,n}(a) \log^\ell b \log^n b_1$$

we can satisfy the boundary condition (3.36) at $a = 1$. Generally speaking, the approximate solution

$$V_{\text{sing}}(a, b) := b^{-k} V_0(a, \log b) + (1-a)^{\frac{1}{2}} \sum_{j=1}^{m} V_j(a, \log b, \log b_1) b^{j-k} b_1^{-j}$$

will not be smooth at the origin $a = 0$, let alone vanish to fourth order. To remedy this problem, we subtract the correction function

$$\tilde{V}(a, b_1) := b_1^{-k} \tilde{V}_0(a, \log b_1) + (1-a)^{\frac{1}{2}} \sum_{j=1}^{m} \tilde{V}_j(a, \log b_1, \log b_1) b_1^{-k}$$

which solves the homogeneous equation $L_a \tilde{V} \in \mathcal{F}_{k+1} + \mathcal{F}_{k+2}$ and has the same singular behavior at $a = 0$ as V_{sing}. More precisely, we first set $b_1 = b$ in $V_{\text{sing}}(a, b)$ and write the resulting expression in the form

$$b^{-k} \sum_\nu V_\nu(a) \log^\nu b_1.$$

In view of our discussion regarding the singularity at $a = 0$, we see that

$$V_\nu(a) = c_\nu \varphi_0(a) + c_\nu' \varphi_1(a) + O\big(a^4\big)$$

where $O(a^4)$ is analytic and φ_1 is the regular homogeneous solution, i.e., $L_a \varphi_1 = 0$, $\varphi_1(a) = a^2(1 + O(a^2))$. Hence, we see that

$$\tilde{V}(a, b_1) := b_1^{-k} \sum_\nu \big(c_\nu \varphi_0(a) + c_\nu' \varphi_1(a)\big) \log^\nu b_1$$

has the desired properties, i.e.,

$$v := V_{\text{sing}} - \tilde{V}$$

vanishes to fourth order at $a = 0$. Finally, as above one checks that

$$L_{ab} \tilde{V} \in \mathcal{Q}_{k+1},$$

which therefore is an error. Finally, the error $f_{k+1} + f_{k+2}$ generated by this entire procedure vanishes at least to second order at the origin as claimed. □

By design, Lemma 3.8 allows for arbitrary many iterations. Therefore, we can now carry out the process leading to v_2 as explained above, see (3.19). At each step we gain a power of b^{-1} or b_1^{-1}, while paying at most one power of $\log b$ and $\log b_1$. We iterate sufficiently often, and let

$$v_2 = w_2 + w_3 + \cdots.$$

By construction v_2 vanishes of order four at $a = 0$, therefore we can factor out an a^4 to obtain

$$v_2 \in a^4 IS(1, \mathcal{Q}_1).$$

Recalling also that we have neglected terms of the form $(\lambda(t)t)^{-2} IS^4(1)$, we find that the remaining error satisfies

$$t^2 e_2 \in a^2 IS^4(1, \mathcal{Q}'_{2\beta}) + IS^4(1, \mathcal{Q}_{2\beta})$$

as desired.

Step 3. We now consider the general setup. *Commence with e_{2k}, $k \geqslant 1$, satisfying* (3.15) *and choose v_{2k+1} so that* (3.12), (3.13) *hold with k replaced by $k+1$.* Note that we can move that part of e_{2k} which belongs to

$$a^2 IS^4\big((\log R)^{k-1}, \mathcal{Q}'_{2\beta k}\big)$$

into the next error, e_{2k+1}. Hence we only need to deal with the part of e_{2k} in

$$IS^4\big((\log R)^{k-1}, \mathcal{Q}_{2\beta k}\big),$$

which we denote as e^0_{2k}. Proceeding as in Step 1, we then set

$$\begin{aligned}(t\lambda)^2 v_{2k+1}(R, a, b, b_1) &= \Theta(R)\int_0^R \Phi(R') t^2 e^0_{2k}(R', a, b, b_1) R'\, dR' \\ &\quad - \Phi(R)\int_1^R \Theta(R') t^2 e^0_{2k}(R', a, b, b_1) R'\, dR'.\end{aligned}$$

Here we treat a, b, b_1 as constant parameters. Then it is clear that

$$v_{2k+1} \in IS^4\big(R^2(\log R)^k, \mathcal{Q}_{2\beta(k+1)}\big).$$

We need to check that the error satisfies (3.13) for $k+1$ instead of k. The error is comprised of the terms arising from ∂_t^2, when one of the variables a, b, b_1 is differentiated, as well as the nonlinear terms. More precisely, we write

$$e_{2k+1} = N_{2k+1}(v_{2k+1}) + E^t v_{2k+1} + E^a v_{2k+1}$$

where the first represents nonlinear errors, the second represents $\partial_t^2 v_{2k+1}$, and the third represents those constituents in

$$\left(-\partial_t^2+\partial_r^2+\frac{1}{r}\partial_r\right)v_{2k+1}(R,a,b,b_1)$$

in which at least one derivative falls on a, or b, or b_1. It is straightforward to check that

$$t^2E^t v_{2k+1}\in IS^4\big(R^2(\log R)^k,\mathcal{Q}_{2\beta(k+1)}\big)\subset IS^4\big(R^2(\log R)^k,\mathcal{Q}'_{2\beta(k+1)}\big).$$

Next, the terms in $t^2E^a v_{2k+1}$ are of the form

$$\begin{gathered}\big[\big(1-a^2\big)\partial_a^2+\big(a^{-1}-2a\big)\partial_a\big]v_{2k+1}(R,a,b,b_1),\\ \big[\big(1-a^2\big)\partial_a+\big(a^{-1}-2a\big)\big]\big(\partial_a b_1\partial_{b_1}v_{2k+1}(R,a,b,b_1)\big),\\ \big(1-a^2\big)t\partial_t aR\partial_R\partial_a v_{2k+1}(R,a,b,b_1)-\big(1-a^2\big)a^{-1}R\partial_a\partial_R v_{2k+1}(R,a,b,b_1).\end{gathered}$$

Each of these is easily seen to be in $IS^4(R^2(\log R)^k,\mathcal{Q}'_{2\beta(k+1)})$. The nonlinear errors are of the form

$$\frac{6}{r^2}\big(u_{2k}^2-Q^2\big)v_{2k+1}+\frac{2}{r^2}\big(3v_{2k+1}^2u_{2k}+v_{2k+1}^3\big).$$

For the term on the left, expand $u_{2k}=Q+\sum_{1\leqslant i\leqslant 2k}v_i$. Using that

$$\sum_{1\leqslant i\leqslant 2k}v_i\in IS^4\big(R^2,\mathcal{Q}_{2\beta}\big),$$

we check that

$$t^2\frac{6}{r^2}\big(u_{2k}^2-Q^2\big)v_{2k+1}\in IS^4\big(R^2(\log R)^k,\mathcal{Q}_{2\beta(k+1)}\big).$$

Similarly, we get

$$\frac{2t^2}{r^2}\big(3v_{2k+1}^2u_{2k}+v_{2k+1}^3\big)\in IS^4\big(R^2(\log R)^k,\mathcal{Q}_{2\beta(k+1)}\big).$$

Step 4. *Commence with e_{2k-1}, $k\geqslant 1$, satisfying* (3.13) *and choose v_{2k} so that* (3.14), (3.15) *hold.* Pick the leading order term in e_{2k-1}, which can be written as

$$t^2e_{2k-1}^0:=R^2\sum_{j=0}^{k-1}g_j(a,b,b_1)(\log R)^j,$$

with $g_j(a)\in\mathcal{Q}'_{2\beta k}$. We then claim that the error $e_{2k-1}^1:=e_{2k-1}-e_{2k-1}^0$ can be absorbed into e_{2k}. Indeed, we can write

$$e_{2k-1}^1=a^2e_{2k-1}^1+\big(1-a^2\big)e_{2k-1}^1,$$

and we have $(1-a^2)\mathcal{Q}'_{2\beta k} \subset \mathcal{Q}_{2\beta k}$. Next, rewrite

$$t^2 e^0_{2k-1} = \sum_{j=0}^{k-1} h_j(a,b,b_1)(\log R)^j, \quad h_j(a,b,b_1) = a^2 g_j(a,b,b_1) \in a^2 \mathcal{Q}'_{2(k-1)\beta}.$$

We first seek an approximate solution w_{2k} for (3.6) of the form

$$w_{2k} = \sum_{j=0}^{k-1} z_j(a,b,b_1)(\log R)^j, \quad z_j \in a^4 \mathcal{Q}_{2(k-1)\beta}.$$

This we then refine, iterating application of Lemma 3.8 sufficiently often to obtain v_{2k}. To find the functions z_j we proceed inductively, starting with the largest power of $\log R$. Indeed, matching corresponding powers of $\log R$, we get a recursive system. Denoting

$$L^\infty := t^2\left(-\partial_t^2 + \partial_r^2 + \frac{1}{r}\partial_r - \frac{4}{r^2}\right)$$

we calculate

$$\begin{aligned}
L^\infty w_{2k}(a,b) &= \sum_{j=0}^{k-1}\biggl\{(\log R)^j L^\infty z_j - 2(t\partial_t)z_j\,(t\partial_t)(\log R)^j + 2(t\partial_r)z_j(t\partial_r)(\log R)^j \\
&\quad + t^2 z_j\left(-\partial_t^2 + \partial_r^2 + \frac{1}{r}\partial_r\right)(\log R)^j\biggr\} \\
&= \sum_{j=0}^{k-1}\bigl\{(\log R)^j L_{ab} z_j + j(\log R)^{j-1} L^1_{ab} z_j + j(j-1)(\log R)^{j-2} L^2_{ab} z_j\bigr\}
\end{aligned}$$

where

$$L^1_{ab} = -2\left(1+\frac{\beta}{b}\right)\left(a\partial_a + \partial_b + \left(1 - \frac{ap'(a)}{p(a)}\right)\partial_{b_1}\right) + 2a^{-1}\left(\partial_a - \frac{p'(a)}{p(a)}\partial_{b_1}\right) - 1 - \frac{\beta}{b} + \frac{\beta}{b^2},$$

$$L^2_{ab} = \left(1+\frac{\beta}{b}\right)^2 + a^{-2}.$$

This leads to the recursive system for $0 \leqslant j \leqslant k-1$,

$$L_{ab} z_j = h_j - (j+1)L^1_{ab} z_{j+1} - (j+1)(j+2)L^2_{ab} z_{j+2}, \qquad z_k = z_{k+1} = 0. \tag{3.37}$$

Since $h_j \in a^2\mathcal{Q}'_{2(k-1)\beta}$ and we seek approximate solutions $z_j \in a^4\mathcal{Q}_{2(k-1)\beta}$, it suffices to take the principal part of the system (3.37), namely

$$L_{ab} z_j = h_j + (j+1)\bigl(1 + 2\bigl(a - a^{-1}\bigr)\partial_a\bigr)z_{j+1} - (j+1)(j+2)\bigl(1 + a^{-2}\bigr)z_{j+2},$$
$$z_k = z_{k+1} = 0.$$

For this we apply Lemma 3.8 to obtain approximate solutions $z_j \in a^4 \mathcal{Q}_{2(k-1)\beta}$ with lower order errors

$$L_{ab} z_j - \big[h_j + (j+1)\big(1 + 2\big(a - a^{-1}\big)\partial_a z_{j+1}\big)\big] \in a^2 \mathcal{Q}'_{2(k-1)\beta+1}.$$

The other terms on the right-hand side of (3.37) have a similar form,

$$(j+1)\big(L^1_{ab} + 1 + 2\big(a - a^{-1}\big)\partial_a\big) z_{j+1} - (j+1)(j+2) L^2_{ab} z_{j+2} \in a^2 \mathcal{Q}'_{2(k-1)\beta+1}.$$

In addition to the above error terms, by adding w_{2k} to the approximate solution we have also generated errors from the nonlinear terms, which we recall are (upon multiplication by t^2)

$$\frac{6t^2}{r^2}\big(u^2_{2k-1} - 1\big) w_{2k} + \frac{2t^2}{r^2}\big(3w^2_{2k} u_{2k-1} + w^3_{2k}\big)$$

where $u_{2k-1} = Q + v_1 + \cdots + v_{2k-1}$. We expand the first term here in the form

$$\frac{t^2}{r^2}(Q - 1 + v_1 + \cdots + v_{2k-1})(Q + 1 + v_1 + \cdots + v_{2k-1}) w_{2k}$$

with $v_1 = v_{10} + v_{11}$. First we write

$$\frac{t^2}{r^2}(Q+1)(Q-1) w_{2k} = a^{-4} \frac{R^2}{1+R^2} \frac{1}{(t\lambda)^2} w_{2k} \in IS^4\big((\log R)^{k-1}, \mathcal{Q}_{2k\beta}\big),$$

which we can absorb into e_{2k}. The terms

$$\frac{t^2}{r^2}(Q \pm 1) v_{10} w_{2k}$$

are similar but more simpler. On the other hand, we recall from Step 1 that v_{11} satisfies $v_{11} \in IS^4(R^2, \mathcal{Q}_{2\beta+1})$. Hence we obtain

$$\frac{t^2}{r^2}(Q-1) v_1 w_{2k} \in a^2 IS^4\big((\log R)^{k-1}, \mathcal{Q}_{2(k-1)\beta+1}\big) \subset IS^4\big((\log R)^{k-1}, \mathcal{Q}'_{2(k-1)\beta+1}\big),$$

which we cannot absorb into e_{2k} yet, whence we iteratively apply the preceding procedure to it. The remaining interactions satisfy at least

$$\frac{t^2}{r^2}(Q \pm 1) v_j w_{2k},\ \frac{t^2}{r^2} v_i v_j w_{2k} \in IS^4\big((\log R)^{k-1}, \mathcal{Q}'_{2(k-1)\beta+2}\big),$$

and we re-iterate the preceding procedure for those which cannot yet be absorbed into e_{2k}. We similarly deduce

$$\frac{2t^2}{r^2}\big(3w^2_{2k} u_{2k-1} + w^3_{2k}\big) \in IS^4\big((\log R)^{k-1}, \mathcal{Q}_{2k\beta}\big),$$

which can therefore be absorbed into e_{2k}. We now re-iterate (sufficiently often) the procedure from the beginning of the present step for those errors which cannot yet be absorbed into e_{2k}, resulting in $w_{2k} = w_{2k}^0, w_{2k}^1, \dots, w_{2k}^{2\beta}$. Finally, $v_{2k} := \sum_{j=0}^{2\beta} w_{2k}^j$ has all the desired properties. $\square$

4. The analysis of the underlying strongly singular Sturm–Liouville operator

In this section we develop the scattering and spectral theory of the linearized operator $\mathcal{L}$. The main tool developed in this section, which is crucial to this paper, is the distorted Fourier transform. The main difference between the linearized operator in [11] and the one of this paper is that in [11] the linearized operator had a zero energy resonance and here zero is an eigenvalue. In both instances, though, there is no negative spectrum (unlike the semi-linear case [12], where we had to deal with a negative eigenvalue and the resulting exponential instabilities).

Definition 4.1. The half-line operator

$$\mathcal{L} := -\frac{d^2}{dR^2} + \frac{15}{4R^2} - \frac{24}{(1+R^2)^2}$$

on $L^2(0,\infty)$ is self-adjoint with domain

$$\mathrm{Dom}(\mathcal{L}) = \big\{ f \in L^2\big((0,\infty)\big):\ f, f' \in AC_{\mathrm{loc}}\big((0,\infty)\big),\ \mathcal{L}f \in L^2\big((0,\infty)\big)\big\}.$$

Because of the strong singularity of the potential at $R=0$ no boundary condition is needed there to insure self-adjointness. Technically speaking, this means that $\mathcal{L}_0$ and $\mathcal{L}$ are in the *limit point case* at $R=0$, see Gesztesy, Zinchenko [5]. We remark that $\mathcal{L}_0$ and $\mathcal{L}$ are in the limit point case at $R=\infty$ by a standard criterion (sub-quadratic growth of the potential).

Lemma 4.2. *The spectrum of $\mathcal{L}$ is purely absolutely continuous and equals* $\mathrm{spec}(\mathcal{L}) = [0,\infty)$.

Proof. That $\mathcal{L}$ has no negative spectrum it follows from

$$\mathcal{L}\phi_0 = 0, \qquad \phi_0(R) = \frac{R^{5/2}}{(1+R^2)^2} \tag{4.1}$$

with ϕ_0 positive (by the Sturm oscillation theorem). The purely absolute continuity of the spectrum of $\mathcal{L}$ is an immediate consequence of the fact that the potential of $\mathcal{L}$ is integrable at infinity. $\square$

We now briefly summarize the results from [5] relevant for our purposes, see Section 3 in their paper, in particular Example 3.10.

Theorem 4.3.

(a) *For each $z \in \mathbb{C}$ there exists a fundamental system $\phi(R,z)$, $\theta(R,z)$ for $\mathcal{L} - z$ which is analytic in z for each $R > 0$ and has the asymptotic behavior*

$$\phi(R,z) \sim R^{\frac{5}{2}}, \qquad \theta(R,z) \sim \frac{1}{4} R^{-\frac{3}{2}} \quad \text{as } R \to 0. \tag{4.2}$$

In particular, their Wronskian is $W(\theta(\cdot,z),\phi(\cdot,z)) = 1$ for all $z \in \mathbb{C}$. We remark that $\phi(\cdot,z)$ is the Weyl–Titchmarsh solution[3] *of $\mathcal{L} - z$ at $R = 0$. By convention, $\phi(\cdot,z), \theta(\cdot,z)$ are real-valued for $z \in \mathbb{R}$.*

(b) *For each $z \in \mathbb{C}$, $\operatorname{Im} z > 0$, let $\psi^+(R,z)$ denote the Weyl–Titchmarsh solution of $\mathcal{L} - z$ at $R = \infty$ normalized so that*

$$\psi^+(R,z) \sim z^{-\frac{1}{4}} e^{iz^{\frac{1}{2}}R} \quad \text{as } R \to \infty, \ \operatorname{Im} z^{\frac{1}{2}} > 0.$$

If $\xi > 0$, then the limit $\psi^+(R,\xi + i0)$ exists pointwise for all $R > 0$ and we denote it by $\psi^+(R,\xi)$. Moreover, define $\psi^-(\cdot,\xi) := \overline{\psi^+(\cdot,\xi)}$. Then $\psi^+(R,\xi)$, $\psi^-(R,\xi)$ form a fundamental system of $\mathcal{L} - \xi$ with asymptotic behavior $\psi^\pm(R,\xi) \sim \xi^{-\frac{1}{4}} e^{\pm i\xi^{\frac{1}{2}}R}$ as $R \to \infty$.

(c) *The spectral measure of $\mathcal{L}$ is given by*

$$\mu(d\xi) = \|\phi_0\|_2^{-2}\delta_0 + \rho(\xi)\,d\xi, \qquad \rho(\xi) := \frac{1}{\pi}\operatorname{Im} m(\xi + i0)\chi_{[\xi>0]} \tag{4.3}$$

with the "generalized Weyl–Titchmarsh" function

$$m(\xi) = \frac{W(\theta(\cdot,\xi),\psi^+(\cdot,\xi))}{W(\psi^+(\cdot,\xi),\phi(\cdot,\xi))}. \tag{4.4}$$

(d) *The distorted Fourier transform defined as*

$$\mathcal{F}: f \to \hat{f}(\xi) = \lim_{b\to\infty}\int_0^b \phi(R,\xi)f(R)\,dR$$

for all $\xi \geqslant 0$ is a unitary operator from $L^2(\mathbb{R}^+)$ to $L^2(\mathbb{R}^+,\mu) = \mathbb{R} \oplus L^2(\mathbb{R}^+,\rho)$ and its inverse is given by

$$\mathcal{F}^{-1}: \hat{f} \to f(R) = \hat{f}(0)\|\phi_0\|_2^{-2}\phi_0(R) + \lim_{s\to\infty}\int_0^s \phi(R,\xi)\hat{f}(\xi)\,\rho(\xi)\,d\xi. \tag{4.5}$$

Here lim *refers to the $L^2(\mathbb{R}^+,\mu)$, respectively the $L^2(\mathbb{R}^+)$, limit.*

Remark 4.4. It is best to view the distorted Fourier transform of any $f \in L^2(\mathbb{R}^+)$ as a vector, namely $f \mapsto \binom{a}{g(\cdot)}$ where $a \in \mathbb{R}$ and $g \in L^2(\mathbb{R}^+,\rho)$. The inversion formula being

$$f = a\|\phi_0\|_2^{-2}\phi_0 + \int_0^\infty \phi(\cdot,\xi)g(\xi)\rho(\xi)\,d\xi.$$

[3] Our $\phi(\cdot,z)$ is the $\tilde{\phi}(z,\cdot)$ function from [5] where the analyticity is only required in a strip around $\mathbb{R}$ – but here there is no need for this restriction.

The first term is the projection of f onto ϕ_0, whereas the second one is the projection onto the orthogonal complement of ϕ_0. We remark that

$$\|\phi_0\|_2^2 = \int_0^\infty \frac{R^5}{(1+R^2)^4}\,dR = \frac{1}{6}.$$

4.1. Asymptotic behavior of ϕ and θ

Beginning with two explicit solutions for $\mathcal{L}f = 0$, namely

$$\phi_0(R) = \frac{R^{\frac{5}{2}}}{(1+R^2)^2}, \qquad \theta_0(R) = \frac{-1 - 8R^2 + 24R^4 \log R + 8R^6 + R^8}{4R^{\frac{3}{2}}(1+R^2)^2}$$

we construct power series expansions for ϕ from (4.2) in $z \in \mathbb{C}$ when $R > 0$ is fixed. A similar expansion is possible for $\theta(R,z)$. Since is it not only more complicated but also not needed here, we skip it.

Proposition 4.5. *For any $z \in \mathbb{C}$ the solution $\phi(R,z)$ from Theorem* 4.3 *admits an absolutely convergent asymptotic expansion*

$$\phi(R,z) = \phi_0(R) + R^{-\frac{3}{2}} \sum_{j=1}^\infty \big(R^2 z\big)^j \tilde{\phi}_j\big(R^2\big).$$

The functions $\tilde{\phi}_j$ are holomorphic in $\Omega = \{\operatorname{Re} u > -\frac{1}{2}\}$ and satisfy the bounds

$$\big|\tilde{\phi}_j(u)\big| \leqslant \frac{C^j}{j!} |u|^2 \langle u\rangle^{-1}, \quad j \geqslant 1,$$

for all $u \in \Omega$. In particular,[4] *in the region $\xi^{-\frac{1}{4}} \ll R \ll \xi^{-\frac{1}{2}}$,*

$$\begin{aligned} \big|\phi(R,\xi)\big| &\asymp R^4 \xi \phi_0(R) \asymp R^{\frac{5}{2}} \xi, \\ \big|\partial_R \phi(R,\xi)\big| &\asymp R^{\frac{3}{2}} \xi. \end{aligned} \tag{4.6}$$

Proof. Write $\phi(R,z) = \sum_{j=0}^\infty z^j \phi_j(R)$. The functions ϕ_j then need to solve $\mathcal{L}\phi_j = \phi_{j-1}$. Since ϕ_0 is not analytic, it is technically convenient to set $\phi_j(R) = R^{-\frac{3}{2}} f_j(R)$ (note that $R^{-\frac{3}{2}}$ is the decay of ϕ_0). Our system of ODEs is then, with $j \geqslant 1$,

$$\mathcal{L}\big(R^{-\frac{3}{2}} f_j\big) = R^{-\frac{3}{2}} f_{j-1}, \qquad f_0(R) = \frac{R^4}{1+R^4}.$$

[4] If $a, b > 0$, then $a \ll b$ means that $a < \varepsilon b$ for some small constant $\varepsilon > 0$, whereas $a \asymp b$ means that for some constant $C > 0$ one has $C^{-1} a < b < Ca$.

The forward fundamental solution for $\mathcal{L}$ is

$$H(R,R') = \big(\phi_0(R)\theta_0(R') - \phi_0(R')\theta_0(R)\big)1_{[R>R']}.$$

Hence we have the iterative relation

$$f_j(R) = \int_0^R R^{\frac{3}{2}}(R')^{-\frac{3}{2}}\big(\phi_0(R)\theta_0(R') - \phi_0(R')\theta_0(R)\big) f_{j-1}(R')\,dR',$$

$$f_0(R) = \frac{R^4}{(1+R^2)^2}.$$

Using the expressions for ϕ_0, θ_0 we rewrite this as

$$f_j(R) = \int_0^R \big[R^4\big(-1-8R'^2+24R'^4\log R'+8R'^6+R'^8\big) \\ - R'^4\big(-1-8R^2+24R^4\log R+8R^6+R^8\big)\big]\frac{f_{j-1}(R')R'}{R'^4(1+R^2)^2(1+R'^2)^2}\,dR'.$$

We claim that all functions f_j extend to even holomorphic functions in any even simply connected domain not containing $\pm i$, vanishing at 0. Indeed, we now suppose that f_{j-1} has these properties and we shall prove them for f_j. Clearly, f_j extends to a holomorphic function in any even simply connected domain not containing $\pm i$ and 0. We first show that at 0 there is at most an isolated singularity. For this we consider a branch of the logarithm which is holomorphic in $\mathbb{C}\setminus\mathbb{R}^-$ and show that $f_j(R+i0) = f_j(R-i0)$ for $R<0$. Disregarding the terms not involving logarithms, we need to show that for any holomorphic function g we have

$$\int_0^{R+i0}\big(\log R' - \log(R+i0)\big)g(R')\,dR' = \int_0^{R-i0}\big(\log R' - \log(R-i0)\big)g(R')\,dR'.$$

This is obvious since for $R'<0$ we have

$$\log(R'+i0) - \log(R+i0) = \log(R'-i0) - \log(R-i0).$$

The singularity at 0 is a removable singularity. Indeed, for R' close to 0 we have $|f_{j-1}(R')| \lesssim |R'|$ which by a crude bound on the denominator in the above integral leads to $|f_j(R)| \lesssim |R|$ (again with R close to 0). This also shows that f_j vanishes at 0 (better bounds will be obtained below). The fact that f_j is even is obvious if we substitute $2\log R'$ and $2\log R$ by $\log R'^2$ respectively $\log R^2$ in the integral. This is allowed since due to the above discussion we can use any branch of the logarithm. Indeed, denoting $\tilde{f}_{j-1}(R'^2) = f_{j-1}(R')$ the change of variable $R'^2 = v$ yields the iterative relation, with $\tilde{f}_0(u) = \frac{u^2}{(1+u)^2}$,

$$
\begin{aligned}
\tilde{f}_j(u) = \int_0^u \big[&u^2\big(-1-8v+12v^2\log v+8v^3+v^4\big) \\
&- v^2\big(-1-8u+12u^2\log u+8u^3+u^4\big)\big]\frac{\tilde{f}_{j-1}(v)}{2v^2(1+u)^2(1+v)^2}\,dv. \qquad (4.7)
\end{aligned}
$$

Next, we obtain bounds on the functions $\tilde{f}_j$. To avoid the singularity at -1 we restrict ourselves to the region $U = \{\operatorname{Re} u > -\frac{1}{2}\}$. We claim that the $\tilde{f}_j$ satisfy the bound

$$
\big|\tilde{f}_j(u)\big| \leqslant \frac{C^j}{j!}|u|^{j+2}\langle u\rangle^{-1}.
$$

The kernel above can be estimated by

$$
\left|\frac{u^2(-1-8v+12v^2\log v+8v^3+v^4)-v^2(-1-8u+12u^2\log u+8u^3+u^4)}{2v^2(1+u)^2(1+v)^2}\right| \leqslant C\frac{|u|^2}{|v|^2}.
$$

We have

$$
\big|\tilde{f}_0(u)\big| \leqslant \frac{|u|^2}{1+|u|^2}
$$

which yields

$$
\big|\tilde{f}_1(u)\big| \leqslant C|u|^2\int_0^{|u|}\frac{1}{1+x^2}\,dx \leqslant C|u|^3\langle u\rangle^{-1}.
$$

From here on we use induction, noting that for $j \geqslant 1$

$$
\big|\tilde{f}_{j+1}(u)\big| \leqslant \frac{C^j}{j!}\int_0^{|u|} x^j\langle x\rangle^{-1}|u|^2\,dx \leqslant \frac{C^{j+1}}{(j+1)!}|u|^{j+3}\langle u\rangle^{-1}.
$$

Finally, note that the functions $\tilde{\phi}_j$ are given by $\tilde{\phi}_j(u) = u^{-j}\tilde{f}_j(u)$ and satisfy the desired pointwise bound.

The statement (4.6) follows from the fact that $|\tilde{\phi}_1(u)| \gtrsim u$ for $u \gg 1$. $\square$

We note that although the above series for ϕ converges for all R, z, we can only use it to obtain various estimates for ϕ in the region $|z|R^2 \lesssim 1$. On the other hand, in the region $\xi R^2 \gtrsim 1$ where $z = \xi > 0$, we will represent ϕ in terms of ψ^+ and use the ψ^+ asymptotic expansion, described in what follows.

4.2. The asymptotic behavior of ψ^+

The following result provides good asymptotics for ψ^+ in the region $R^2\xi \gtrsim 1$.

Proposition 4.6. *For any $\xi > 0$, the solution $\psi^+(\cdot, \xi)$ from Theorem 4.3 is of the form*

$$\psi^+(R,\xi) = \xi^{-\frac{1}{4}} e^{iR\xi^{\frac{1}{2}}} \sigma\big(R\xi^{\frac{1}{2}}, R\big), \quad R^2\xi \gtrsim 1$$

where σ admits the asymptotic series approximation

$$\sigma(q,R) \approx \sum_{j=0}^{\infty} q^{-j} \psi_j^+(R), \qquad \psi_0^+ = 1, \qquad \psi_1^+ = \frac{15i}{8} + O\left(\frac{1}{1+R^2}\right)$$

with zero order symbols $\psi_j^+(R)$ that are analytic at infinity,

$$\sup_{R>0} \big|(R\partial_R)^k \psi_j^+(R)\big| < \infty$$

in the sense that for all large integers j_0, and all indices α, β, we have

$$\sup_{R>0} \left| (R\partial_R)^\alpha (q\partial_q)^\beta \left[\sigma(q,R) - \sum_{j=0}^{j_0} q^{-j} \psi_j^+(R) \right] \right| \leqslant c_{\alpha,\beta,j_0} q^{-j_0-1}$$

for all $q > 1$.

Proof. With the notation

$$\sigma(q,R) = \xi^{\frac{1}{4}} \psi^+(R,\xi) e^{-iR\xi^{\frac{1}{2}}}$$

we need to solve the conjugated equation

$$\left(-\partial_R^2 - 2i\xi^{\frac{1}{2}}\partial_R + \frac{15}{4R^2} - \frac{24}{(1+R^2)^2}\right)\sigma\big(R\xi^{\frac{1}{2}}, R\big) = 0. \tag{4.8}$$

We look for a formal power series solving this equation, i.e.,

$$\sigma(q,R) = \sum_{j=0}^{\infty} \xi^{-\frac{j}{2}} f_j(R). \tag{4.9}$$

This yields a recurrence relation for the f_j's,

$$2if_j'(R) = \left(-\frac{d^2}{dR^2} + \frac{15}{4R^2} - \frac{24}{(1+R^2)^2}\right) f_{j-1}(R), \qquad f_0 = 1$$

which is solved by

$$f_j(R) = \frac{i}{2} f'_{j-1}(R) + \frac{i}{2} \int_R^\infty \left(\frac{15}{4R'^2} - \frac{24}{(1+R'^2)^2} \right) f_{j-1}(R')\, dR'.$$

Extending this into the complex domain, it is easy to see that the functions f_j are holomorphic in $\mathbb{C} \setminus [-i, i]$. They are also holomorphic at ∞, and the leading term in the Taylor series at ∞ is R^{-j}. At 0 one has the estimate

$$\left|(R\partial_R)^k f_j(R)\right| \leqslant c_{jk} R^{-j} \quad \forall R > 0$$

which is easy to establish inductively. The functions

$$\psi_j^+(R) := R^j f_j(R)$$

now satisfy the desired bounds due to the bounds above on f_j. The remainder of the proof is the same as in our wave-map paper [11] and we skip it. □

4.3. The spectral measure

We now describe the spectral measure by means of (4.4). This requires relating the functions ϕ, θ and $\psi^\pm$. By examining the asymptotics at $R = 0$ we see that

$$W(\theta, \phi) = 1. \tag{4.10}$$

Also by examining the asymptotics as $R \to \infty$ we obtain

$$W(\psi^+, \psi^-) = -2i. \tag{4.11}$$

Lemma 4.7.

(a) *We have*

$$\phi(R, \xi) = a(\xi)\psi^+(R, \xi) + \overline{a(\xi)\psi^+(R, \xi)} \tag{4.12}$$

where a is smooth, always nonzero, and has size

$$\left|a(\xi)\right| \asymp \begin{cases} 1 & \text{if } \xi \ll 1, \\ \xi^{-1} & \text{if } \xi \gtrsim 1. \end{cases}$$

Moreover, it satisfies the symbol type bounds

$$\left|(\xi\partial_\xi)^k a(\xi)\right| \leqslant c_k \left|a(\xi)\right| \quad \forall \xi > 0.$$

(b) *The absolutely continuous part of the spectral measure $\mu(d\xi)$ has density $\rho(\xi)$ which satisfies*

$$\rho(\xi) \asymp \begin{cases} 1 & \text{if } \xi \ll 1, \\ \xi^2 & \text{if } \xi \gtrsim 1 \end{cases}$$

with symbol type estimates on the derivatives.

Proof. (a) Since ϕ is real-valued, due to (4.11), the relation (4.12) above holds with

$$a(\xi) = -\frac{i}{2} W\big(\phi(\cdot,\xi), \psi^-(\cdot,\xi)\big).$$

We evaluate the Wronskian in the region where both the $\psi^+(R,\xi)$ and $\phi(R,\xi)$ asymptotics are useful, i.e., where $R^2\xi \approx 1$. The bounds from above on a and its derivatives thus follow from Propositions 4.5 and 4.6.

For the bound from below on a we use that

$$\big|a(\xi)\big| \geqslant \frac{|\partial_R \phi(R,\xi)|}{2|\partial_R \psi^+(R,\xi)|}$$

which was obtained in [11]. We use this relation for $R = \delta\xi^{-\frac{1}{2}}$ with a small constant δ. Then by Proposition 4.5 we have

$$\big|\partial_R \phi(R,\xi)\big| \gtrsim \begin{cases} R^{-\frac{1}{2}}, & \xi \ll 1, \\ R^{\frac{3}{2}}, & \xi \gtrsim 1 \end{cases}$$

while by Proposition 4.6

$$\big|\partial_R \psi^+(R,\xi)\big| \lesssim \xi^{\frac{1}{4}}.$$

This give the desired bound from below on a.

(b) In [11] it was shown that

$$\rho(\xi) = \frac{1}{\pi}\big|a(\xi)\big|^{-2}.$$

The bounds on $\rho(\xi)$ now follow from part (a). □

5. The transference identity

We now write the radiation part $\tilde{\varepsilon}$ in terms of the generalized Fourier basis $\phi(R,\xi)$ from Theorem 4.3, i.e.,

$$\tilde{\varepsilon}(\tau, R) = x_0(\tau)\phi_0(R) + \int_0^\infty x(\tau,\xi)\phi(R,\xi)\rho(\xi)\, d\xi.$$

As in [11,12] we define the error operator $\mathcal{K}$ by

$$\widehat{R\,\partial_R u} = -2\xi\,\partial_\xi \hat{u} + \mathcal{K}\hat{u} \tag{5.1}$$

where the hat denotes the "distorted Fourier transform" and the operator $-2\xi\,\partial_\xi$ acts only on the continuous part of the spectrum. In view of Remark 4.4 we obtain a matrix representation for $\mathcal{K}$, namely

$$\mathcal{K} = \begin{pmatrix} \mathcal{K}_{\mathrm{ee}} & \mathcal{K}_{\mathrm{ec}} \\ \mathcal{K}_{\mathrm{ce}} & \mathcal{K}_{\mathrm{cc}} \end{pmatrix}.$$

Here 'c' and 'e' stand for "continuous" and "eigenvalue," respectively. Using the expressions for the direct and inverse Fourier transform in Theorem 4.3 we obtain

$$\begin{aligned}
\mathcal{K}_{\mathrm{ee}} &= \langle R\partial_R\phi_0(R), \phi_0(R)\rangle_{L^2_R}, \\
\mathcal{K}_{\mathrm{ec}} f &= \left\langle \int_0^\infty f(\xi) R\partial_R\phi(R,\xi)\rho(\xi)\,d\xi, \phi_0(R)\right\rangle_{L^2_R}, \\
\mathcal{K}_{\mathrm{ce}}(\eta) &= \langle R\partial_R\phi_0(R), \phi(R,\eta)\rangle_{L^2_R}, \\
\mathcal{K}_{\mathrm{cc}} f(\eta) &= \left\langle \int_0^\infty f(\xi) R\partial_R\phi(R,\xi)\rho(\xi)\,d\xi, \phi(R,\eta)\right\rangle_{L^2_R} \\
&\quad + \left\langle \int_0^\infty 2\xi\,\partial_\xi f(\xi)\phi(R,\xi)\rho(\xi)\,d\xi, \phi(R,\eta)\right\rangle_{L^2_R}.
\end{aligned} \tag{5.2}$$

Integrating by parts with respect to R in the first two relations we obtain

$$\mathcal{K}_{\mathrm{ee}} = -\frac{1}{2}\|\phi_0\|_2^2 = -\frac{1}{12}, \qquad \mathcal{K}_{\mathrm{ec}} f = -\int_0^\infty f(\xi) K_{\mathrm{e}}(\xi)\rho(\xi)\,d\xi, \qquad \mathcal{K}_{\mathrm{ce}}(\eta) = K_{\mathrm{e}}(\eta)$$

where

$$K_{\mathrm{e}}(\eta) = \langle R\partial_R\phi_0(R), \phi(R,\eta)\rangle_{L^2_R}.$$

Integrating by parts with respect to ξ in (5.2) yields

$$\begin{aligned}
\mathcal{K}_{\mathrm{cc}} f(\eta) &= \left\langle \int_0^\infty f(\xi)[R\partial_R - 2\xi\,\partial_\xi]\phi(R,\xi)\rho(\xi)\,d\xi, \phi(R,\eta)\right\rangle_{L^2_R} \\
&\quad - 2\left(1 + \frac{\eta\rho'(\eta)}{\rho(\eta)}\right) f(\eta)
\end{aligned} \tag{5.3}$$

where the scalar product is to be interpreted in the principal value sense with $f \in C_0^\infty((0,\infty))$.

In this section, we study the boundedness properties of the operator $\mathcal{K}$. We begin with a description of the function K_{e} and of the kernel $K_0(\eta,\xi)$ of $\mathcal{K}_{\mathrm{cc}}$.

Theorem 5.1.

(a) *The operator $\mathcal{K}_{\mathrm{cc}}$ can be written as*

$$\mathcal{K}_{\mathrm{cc}} = -\left(\frac{3}{2} + \frac{\eta\rho'(\eta)}{\rho(\eta)}\right)\delta(\xi-\eta) + \mathcal{K}_0 \tag{5.4}$$

where the operator $\mathcal{K}_0$ has a kernel $K_0(\eta,\xi)$ of the form[5]

$$K_0(\eta,\xi) = \frac{\rho(\xi)}{\eta-\xi}F(\xi,\eta) \tag{5.5}$$

with a symmetric function $F(\xi,\eta)$ of class C^2 in $(0,\infty)\times(0,\infty)$ and continuous on $[0,\infty)^2$. Moreover, it satisfying the bounds

$$\begin{aligned}
\left|F(\xi,\eta)\right| &\lesssim \begin{cases} \xi+\eta, & \xi+\eta\leqslant 1,\\ (\xi+\eta)^{-\frac{5}{2}}(1+|\xi^{\frac12}-\eta^{\frac12}|)^{-N}, & \xi+\eta\geqslant 1,\end{cases}\\
\left|\partial_\xi F(\xi,\eta)\right| + \left|\partial_\eta F(\xi,\eta)\right| &\lesssim \begin{cases} 1, & \xi+\eta\leqslant 1,\\ (\xi+\eta)^{-3}(1+|\xi^{\frac12}-\eta^{\frac12}|)^{-N}, & \xi+\eta\geqslant 1,\end{cases}\\
\sup_{j+k=2}\left|\partial_\xi^j\partial_\eta^k F(\xi,\eta)\right| &\lesssim \begin{cases} (\xi+\eta)^{-1}, & \xi+\eta\leqslant 1,\\ (\xi+\eta)^{-\frac{7}{2}}(1+|\xi^{\frac12}-\eta^{\frac12}|)^{-N}, & \xi+\eta\geqslant 1\end{cases}
\end{aligned}$$

where N is an arbitrary large integer.

(b) *The function K_{e} and K_{e}' are bounded, continuous, and rapidly decaying at infinity.*

Proof. We first establish the off-diagonal behavior of $\mathcal{K}_{\mathrm{cc}}$, and later return to the issue of identifying the δ-measure that sits on the diagonal. We begin with (5.3) with $f \in C_0^\infty((0,\infty))$. The integral

$$u(R) = \int_0^\infty f(\xi)[R\partial_R - 2\xi\partial_\xi]\phi(R,\xi)\rho(\xi)\,d\xi$$

behaves like $R^{\frac52}$ at 0 and is a Schwartz function at infinity. The second factor $\phi(R,\eta)$ in (5.3) also decays like $R^{\frac52}$ at 0 but at infinity it is only bounded with bounded derivatives. Then the following integration by parts is justified:

$$\eta\mathcal{K}_{\mathrm{cc}}f(\eta) = \langle u(R), \mathcal{L}\phi(R,\eta)\rangle_{L^2_R} = \langle \mathcal{L}u(R), \phi(R,\eta)\rangle_{L^2_R}.$$

[5] The kernel below is interpreted in the principal value sense.

Moreover,

$$\begin{aligned}(\mathcal{L}u)(R) &= \int_0^\infty f(\xi)[\mathcal{L}, R\partial_R]\phi(R,\xi)\rho(\xi)\,d\xi + \int_0^\infty f(\xi)(R\partial_R - 2\xi\partial_\xi)\xi\phi(R,\xi)\rho(\xi)\,d\xi \\ &= \int_0^\infty f(\xi)[\mathcal{L}, R\partial_R]\phi(R,\xi)\rho(\xi)\,d\xi + \int_0^\infty \xi f(\xi)(R\partial_R - 2\xi\partial_\xi)\phi(R,\xi)\rho(\xi)\,d\xi \\ &\quad - 2\int_0^\infty \xi f(\xi)\phi(R,\xi)\rho(\xi)\,d\xi\end{aligned}$$

with the commutator

$$[\mathcal{L}, R\partial_R] = 2\mathcal{L} + \frac{48}{(1+R^2)^2} - \frac{96R^2}{3(1+R^2)^3} =: 2\mathcal{L} + U(R).$$

Thus,

$$(\mathcal{L}u)(R) = \int_0^\infty f(\xi)U(R)\phi(R,\xi)\rho(\xi)\,d\xi + \int_0^\infty \xi f(\xi)(R\partial_R - 2\xi\partial_\xi)\phi(R,\xi)\rho(\xi)\,d\xi.$$

Hence we obtain

$$\eta\mathcal{K}_{\mathrm{cc}} f(\eta) - \mathcal{K}_{\mathrm{cc}}(\xi f)(\eta) = \left\langle \int_0^\infty f(\xi)U(R)\phi(R,\xi)\rho(\xi)\,d\xi, \phi(R,\eta)\right\rangle_{L^2_R}.$$

The double integral on the right-hand side is absolutely convergent, therefore we can change the order of integration to obtain

$$(\eta-\xi)K_0(\eta,\xi) = \rho(\xi)\big\langle U(R)\phi(R,\xi), \phi(R,\eta)\big\rangle_{L^2_R}.$$

This leads to the representation in (5.5) when $\xi \neq \eta$ with

$$F(\xi,\eta) = \big\langle U(R)\phi(R,\xi), \phi(R,\eta)\big\rangle_{L^2_R}.$$

It remains to study its size and regularity. First, due to our pointwise bound from the previous section,

$$\begin{aligned}\big|\phi(R,\xi)\big| &\lesssim \min\big(R^{\frac52}\langle R\rangle^{-4}\big(1+R^4\xi\big), \xi^{-\frac14}\big) \quad \forall 0 \leqslant \xi < 1, \\ \big|\phi(R,\xi)\big| &\lesssim \min\big(R^{\frac52}, \xi^{-\frac54}\big) \quad \forall \xi > 1.\end{aligned}$$

Note that these bounds imply that for all $\xi \geqslant 0$,

$$\langle R\rangle^{-2}\big|\phi(R,\xi)\big| \lesssim \phi_0(R) \lesssim \langle R\rangle^{-\frac{3}{2}}.$$

Hence, $|F(\xi,\eta)| \lesssim 1$ for all $0 \leqslant \xi, \eta < 1$. Moreover, $F(\xi,\eta)$ is continuous on $[0,\infty)^2$ by dominated convergence. Finally, using that $|\phi(R,\xi)| \lesssim \xi^{-\frac{5}{4}}$ when $\xi > 1$ implies that

$$\big|F(\xi,\eta)\big| \lesssim \langle\xi\rangle^{-\frac{5}{4}}\langle\eta\rangle^{-\frac{5}{4}} \quad \forall \xi,\eta \geqslant 0. \tag{5.6}$$

We shall improve on this in a number of ways, but first we consider derivatives. By the previous section,

$$\big|\partial_\xi \phi(R,\xi)\big| \lesssim \min\big(R^{\frac{9}{2}}, R\xi^{-\frac{7}{4}}\big) \quad \forall \xi > 1,$$
$$\big|\partial_\xi \phi(R,\xi)\big| \lesssim \min\big(R^{\frac{5}{2}}, R\xi^{-\frac{3}{4}}\big) \quad \forall 0 < \xi < 1.$$

Consequently, if $0 < \xi, \eta < 1$, then

$$\begin{aligned}\big|\partial_\xi F(\xi,\eta)\big| &\lesssim \int_0^\infty \langle R\rangle^{-4}\min\big(R^{\frac{5}{2}}, R\xi^{-\frac{3}{4}}\big)\min\big(\langle R\rangle^{-\frac{3}{2}}\big(1+R^4\eta\big), \eta^{-\frac{1}{4}}\big)\,dR \\ &\lesssim \int_0^{\eta^{-\frac{1}{2}}} \langle R\rangle^{-3}\big(1+R^4\eta\big)\,dR + \int_{\eta^{-\frac{1}{2}}}^\infty \langle R\rangle^{-\frac{3}{2}}\eta^{-\frac{1}{4}}\,dR \lesssim 1\end{aligned}$$

whereas if $0 < \xi < 1 < \eta$, then

$$\big|\partial_\xi F(\xi,\eta)\big| \lesssim \int_0^\infty \langle R\rangle^{-\frac{3}{2}}\eta^{-\frac{5}{4}}\,dR \lesssim \eta^{-\frac{5}{4}}.$$

If $0 < \eta < 1 < \xi$, then

$$\begin{aligned}\big|\partial_\xi F(\xi,\eta)\big| &\lesssim \int_0^\infty \langle R\rangle^{-4}\min\big(R^{\frac{9}{2}}, R\xi^{-\frac{7}{4}}\big)\min\big(\langle R\rangle^{-\frac{3}{2}}\big(1+R^4\eta\big), \eta^{-\frac{1}{4}}\big)\,dR \\ &\lesssim \xi^{-\frac{7}{4}}\int_0^{\eta^{-\frac{1}{2}}} \langle R\rangle^{-\frac{9}{2}}\big(1+R^4\eta\big)\,dR + \int_{\eta^{-\frac{1}{2}}}^\infty \langle R\rangle^{-4}R\xi^{-\frac{7}{4}}\eta^{-\frac{1}{4}}\,dR \lesssim \xi^{-\frac{7}{4}}.\end{aligned}$$

Finally, for $1 < \xi, \eta$,

$$\left|\partial_\xi F(\xi,\eta)\right| \lesssim \int_0^\infty \langle R\rangle^{-4} R \xi^{-\frac{7}{4}} \eta^{-\frac{5}{4}}\, dR \lesssim \xi^{-\frac{7}{4}} \eta^{-\frac{5}{4}}.$$

To summarize,

$$\left|\partial_\xi F(\xi,\eta)\right| \lesssim \langle\xi\rangle^{-\frac{7}{4}} \langle\eta\rangle^{-\frac{5}{4}}, \qquad \left|\partial_\eta F(\xi,\eta)\right| \lesssim \langle\xi\rangle^{-\frac{5}{4}} \langle\eta\rangle^{-\frac{7}{4}} \quad \forall \xi,\eta \geqslant 0. \tag{5.7}$$

For the second derivatives we use that

$$\begin{aligned} \left|\partial_\xi^2 \phi(R,\xi)\right| &\lesssim \min\bigl(R^{\frac{13}{2}}, R^2 \xi^{-\frac{9}{4}}\bigr) \quad \forall \xi > 1, \\ \left|\partial_\xi^2 \phi(R,\xi)\right| &\lesssim \min\bigl(R^{\frac{9}{2}}, R^2 \xi^{-\frac{5}{4}}\bigr) \quad \forall 0 < \xi < 1 \end{aligned}$$

which imply the bounds we always have the estimates

$$\begin{aligned} \left|\partial_{\xi\eta}^2 F(\xi,\eta)\right| &\lesssim \xi^{-\frac{7}{4}} \eta^{-\frac{7}{4}} \quad \forall \xi > 1,\ \eta > 1, \\ \left|\partial_\xi^2 F(\xi,\eta)\right| &\lesssim \xi^{-\frac{9}{4}} \eta^{-\frac{5}{4}} \quad \forall \xi > 1,\ \eta > 1, \\ \left|\partial_\eta^2 F(\xi,\eta)\right| &\lesssim \xi^{-\frac{5}{4}} \eta^{-\frac{9}{4}} \quad \forall \xi > 1,\ \eta > 1. \end{aligned} \tag{5.8}$$

The bounds (5.6), (5.7), and (5.8) are only useful when ξ and η are very close. To improve on them, we consider two cases:

Case 1. $1 \lesssim \xi + \eta$. To capture the cancellations when ξ and η are separated we resort to another integration by parts,

$$\eta F(\xi,\eta) = \bigl\langle U(R)\phi(R,\xi), \mathcal{L}\phi(R,\eta)\bigr\rangle = \bigl\langle \bigl[\mathcal{L}, U(R)\bigr]\phi(R,\xi), \phi(R,\eta)\bigr\rangle + \xi F(\xi,\eta). \tag{5.9}$$

Hence, evaluating the commutator,

$$(\eta - \xi) F(\xi,\eta) = -\bigl\langle \bigl(2U_R(R)\partial_R + U_{RR}(R)\bigr)\phi(R,\xi), \phi(R,\eta)\bigr\rangle. \tag{5.10}$$

Since $U_R(0) = 0$ it follows that $(2U_R(R)\partial_R + U_{RR}(R))\phi(R,\xi)$ vanishes at the same rate as $\phi(R,\xi)$ at $R = 0$. Then we can repeat the argument above to obtain

$$(\eta - \xi)^2 F(\xi,\eta) = -\bigl\langle [\mathcal{L}, 2U_R\partial_R + U_{RR}]\phi(R,\xi), \phi(R,\eta)\bigr\rangle.$$

The second commutator has the form, with $V(R) := -24(1 + R^2)^{-2}$,

$$[\mathcal{L}, 2U_R\partial_R + U_{RR}] = 4U_{RR}\mathcal{L} - 4U_{RRR}\partial_R - U_{RRRR} - 2U_R V_R - 4U_{RR} V.$$

Since $V(R)$, $U(R)$ are even, this leads to

$$(\eta - \xi)^2 F(\xi,\eta) = \bigl\langle \bigl(U^{\mathrm{odd}}(R)\partial_R + U^{\mathrm{even}}(R) + \xi U^{\mathrm{even}}(R)\bigr)\phi(R,\xi), \phi(R,\eta)\bigr\rangle$$

where by U^{odd}, respectively U^{even}, we have generically denoted odd, respectively even, nonsingular rational functions with good decay at infinity. Inductively, one now verifies the identity

$$(\eta-\xi)^{2k}F(\xi,\eta)=\left\langle\left(\sum_{j=0}^{k-1}\xi^j U^{\mathrm{odd}}_{kj}(R)\,\partial_R+\sum_{\ell=0}^{k}\xi^\ell U^{\mathrm{even}}_{k\ell}(R)\right)\phi(R,\xi),\phi(R,\eta)\right\rangle,$$
$$\langle R\rangle\big|U^{\mathrm{odd}}_{kj}(R)\big|+\big|U^{\mathrm{even}}_{k\ell}(R)\big|\lesssim\langle R\rangle^{-4-2k}\quad\forall j,\ell. \tag{5.11}$$

By means of the pointwise bounds on ϕ from above as well as

$$\big|\partial_R\phi(R,\xi)\big|\lesssim\begin{cases}\max(\langle R\rangle^{-\frac12},\xi^{\frac14})\leqslant 1 & \text{if } 0\leqslant\xi\leqslant 1,\\ \min(R^{\frac32},\xi^{-\frac34})\leqslant\xi^{-\frac34} & \text{if } \xi\geqslant 1\end{cases}$$

we infer from this that

$$\big|F(\xi,\eta)\big|\lesssim\frac{\langle\xi\rangle^{k-\frac54}\langle\eta\rangle^{-\frac54}}{(\eta-\xi)^{2k}}\quad\forall\xi,\ \eta>0.$$

Combining this estimate with (5.6) yields, for arbitrary N,

$$\big|F(\xi,\eta)\big|\lesssim(\xi+\eta)^{-\frac52}\big(1+|\xi^{\frac12}-\eta^{\frac12}|\big)^{-N}\quad\text{provided } \xi+\eta\gtrsim 1, \tag{5.12}$$

as claimed. For the derivatives of F we follow a similar procedure. If ξ and η are comparable, then from (5.7), $|\partial_\eta F(\xi,\eta)|\lesssim\langle\xi\rangle^{-3}$. We will use this bound only when $|\xi^{\frac12}-\eta^{\frac12}|<1$ which of course implies that $\xi\asymp\eta\gtrsim 1$. Thus, we now assume that $|\xi^{\frac12}-\eta^{\frac12}|\geqslant 1$ which is the same as $|\xi-\eta|>(\xi+\eta)^{\frac12}$. In this case, we differentiate with respect to η in (5.11). This yields

$$(\eta-\xi)^{2k}\partial_\eta F(\xi,\eta)=\left\langle\left(\sum_{j=0}^{k-1}\xi^j\,U^{\mathrm{odd}}_{kj}(R)\,\partial_R+\sum_{\ell=0}^{k}\xi^\ell U^{\mathrm{even}}_{k\ell}(R)\right)\phi(R,\xi),\,\partial_\eta\phi(R,\eta)\right\rangle$$
$$-2k(\eta-\xi)^{2k-1}F(\xi,\eta).$$

Using the bound on F from (5.12) as well as the usual estimate on $\partial_\eta\phi(R,\eta)$, leads to

$$\big|F(\xi,\eta)\big|\lesssim(\xi+\eta)^{-3}\big(1+\big|\xi^{\frac12}-\eta^{\frac12}\big|\big)^{-N}\quad\text{provided } \xi+\eta\gtrsim 1. \tag{5.13}$$

The second order derivatives with respect to ξ and η are treated in an analogous manner. We note that it is important here that the decay of U^{odd}_{kj} and $U^{\mathrm{even}}_{k\ell}$ improves with k. This is because the optimal second derivative bound for small η, viz. $|\partial_\eta\phi(R,\eta)|\lesssim R^{\frac92}$, has a sizeable growth in R.

Case 2. $\xi,\eta\ll 1$. First, we note that

$$F(0,0)=\langle U\phi_0,\phi_0\rangle=\big\langle\big([\mathcal{L},R\partial_R]-2\mathcal{L}\big)\phi_0,\phi_0\big\rangle=0.$$

Together with the derivative bound (5.7), this implies that

$$\left|F(\xi,\eta)\right| \lesssim \xi+\eta,$$

as claimed. To bound the second order derivatives of F we recall the pointwise bounds, for $0<\xi<1$,

$$\left|\partial_\xi \phi(R,\xi)\right| \lesssim \min\bigl(R^{\frac{5}{2}}, R\xi^{-\frac{3}{4}}\bigr).$$

If $0<\xi<\eta<1$, then these bounds imply that

$$\begin{aligned}\left|\partial_{\xi\eta}F(\xi,\eta)\right| &\lesssim \int_0^{\eta^{-\frac{1}{2}}} \langle R\rangle^{-4}R^5\,dR + \int_{\eta^{-\frac{1}{2}}}^{\xi^{-\frac{1}{2}}} \langle R\rangle^{-4}R^{\frac{7}{2}}\eta^{-\frac{3}{4}}\,dR + \int_{\xi^{-\frac{1}{2}}}^{\infty} \langle R\rangle^{-2}(\xi\eta)^{-\frac{3}{4}}\,dR \\ &\lesssim \eta^{-1}+\xi^{-\frac{1}{4}}\eta^{-\frac{3}{4}}. \end{aligned} \tag{5.14}$$

This bound is only acceptable as long as ξ and η are comparable. Otherwise, if $0<\xi\ll\eta\leqslant 1$, then one needs to exploit the oscillations of $\partial_\eta\phi(R,\eta)$ in the regime $R^2\eta>1$ as provided by Proposition 4.6 and Lemma 4.7. Thus, write

$$\begin{aligned}\partial_\eta\phi(R,\eta) &= \partial_\eta\bigl[a(\eta)\psi^+(R,\eta)+\overline{a(\eta)\psi^+(R,\eta)}\bigr] = 2\,\mathrm{Re}\,\partial_\eta\bigl[a(\eta)\eta^{-\frac{1}{4}}e^{iR\eta^{\frac{1}{2}}}\sigma\bigl(R\eta^{\frac{1}{2}},R\bigr)\bigr] \\ &= 2\,\mathrm{Re}\bigl[\bigl(a(\eta)\eta^{-\frac{1}{4}}\bigr)'e^{iR\eta^{\frac{1}{2}}}\sigma\bigl(R\eta^{\frac{1}{2}},R\bigr)\bigr] + R\,\mathrm{Re}\bigl[ia(\eta)\eta^{-\frac{3}{4}}e^{iR\eta^{\frac{1}{2}}}\sigma\bigl(R\eta^{\frac{1}{2}},R\bigr)\bigr] \\ &\quad + R\,\mathrm{Re}\bigl[a(\eta)\eta^{-\frac{3}{4}}e^{iR\eta^{\frac{1}{2}}}\sigma_q\bigl(R\eta^{\frac{1}{2}},R\bigr)\bigr].\end{aligned}$$

Therefore,

$$\left|\int_{\eta^{-\frac{1}{2}}}^{\infty} U(R)\partial_\xi\phi(R,\xi)\partial_\eta\phi(R,\eta)\,dR\right|$$

$$\lesssim \left|\int_{\eta^{-\frac{1}{2}}}^{\infty} U(R)\partial_\xi\phi(R,\xi)\bigl(a(\eta)\eta^{-\frac{1}{4}}\bigr)'e^{iR\eta^{\frac{1}{2}}}\sigma\bigl(R\eta^{\frac{1}{2}},R\bigr)\,dR\right| \tag{5.15}$$

$$+\left|\int_{\eta^{-\frac{1}{2}}}^{\infty} RU(R)\partial_\xi\phi(R,\xi)a(\eta)\eta^{-\frac{5}{4}}\sigma\bigl(R\eta^{\frac{1}{2}},R\bigr)\partial_R e^{iR\eta^{\frac{1}{2}}}\,dR\right| \tag{5.16}$$

$$+\left|\int_{\eta^{-\frac{1}{2}}}^{\infty} RU(R)\partial_\xi\phi(R,\xi)a(\eta)\eta^{-\frac{5}{4}}\sigma_q\bigl(R\eta^{\frac{1}{2}},R\bigr)\partial_R e^{iR\eta^{\frac{1}{2}}}\,dR\right|. \tag{5.17}$$

The term on the right-hand side of (5.15) is bounded by

$$\int_{\eta^{-\frac{1}{2}}}^{\infty} R^{-\frac{3}{2}}\eta^{-\frac{5}{4}}\,dR \lesssim \eta^{-1}$$

whereas (5.16) and (5.17) require integrating by parts. It will suffice to consider the former. Using that $|\partial_{R\xi}\phi(R,\xi)| \lesssim \min(R^{\frac{3}{2}}, R\xi^{-\frac{1}{4}})$ and that $|\partial_q\sigma(q,R)| \lesssim R^{-1}$, we obtain

$$\begin{aligned}
&\left|\int_{\eta^{-\frac{1}{2}}}^{\infty} RU(R)\partial_\xi\phi(R,\xi)a(\eta)\eta^{-\frac{5}{4}}\sigma\big(R\eta^{\frac{1}{2}},R\big)\partial_R e^{iR\eta^{\frac{1}{2}}}\,dR\right| \\
&\quad\lesssim \eta^{-\frac{5}{4}}\big|RU(R)\partial_\xi\phi(R,\xi)\big|_{R=\eta^{-\frac{1}{2}}} \\
&\qquad + \eta^{-\frac{5}{4}}\int_{\eta^{-\frac{1}{2}}}^{\infty}\big[\langle R\rangle^{-4}\big|\partial_\xi\phi(R,\xi)\big| + \langle R\rangle^{-3}\big|\partial_{R\xi}\phi(R,\xi)\big|\big]\,dR \lesssim \eta^{-1}.
\end{aligned}$$

In conclusion, for all $0 \leqslant \xi, \eta \leqslant 1$,

$$\big|\partial_{\xi\eta}F(\xi,\eta)\big| \lesssim (\xi+\eta)^{-1}$$

as desired. Next, consider $\partial_\xi^2 F(\xi,\eta)$. The bound

$$\big|\partial_\xi^2 F(\xi,\eta)\big| \lesssim \int_0^\infty \langle R\rangle^{-4}\min\big(R^{\frac{9}{2}}, R^2\xi^{-\frac{5}{4}}\big)\langle R\rangle^{\frac{1}{2}}\,dR \lesssim \xi^{-1}$$

is acceptable as long as $0 < \eta \lesssim \xi \leqslant 1$. If, on the other hand, $0 < \xi \ll \eta \leqslant 1$, then differentiating in (5.10) we obtain

$$(\eta-\xi)\partial_\xi^2 F(\xi,\eta) = 2\partial_\xi F(\xi,\eta) - \big\langle \partial_\xi^2\phi(R,\xi), (2U_R\partial_R + U_{RR})\phi(R,\eta)\big\rangle$$

which implies that

$$\begin{aligned}
\big|\partial_\xi^2 F(\xi,\eta)\big| &\lesssim \eta^{-1}\big[\big|\partial_\xi F(\xi,\eta)\big| + \big|\big\langle\partial_\xi^2\phi(R,\xi), U_{RR}\phi(R,\eta)\big\rangle\big| \\
&\quad + \big|\big\langle\partial_\xi^2\phi(R,\xi), R^{-1}U_R R\partial_R\phi(R,\eta)\big\rangle\big|\big].
\end{aligned}$$

The first term in brackets is $\lesssim 1$, the second is bounded by

$$\int_0^{\eta^{-\frac{1}{2}}} \langle R\rangle^{-6}\langle R\rangle^{\frac{9}{2}}\langle R\rangle^{-\frac{3}{2}}\big(1+R^4\eta\big)\,dR + \int_{\eta^{-\frac{1}{2}}}^{\infty} R^{-6}R^{\frac{9}{2}}\eta^{-\frac{1}{4}}\,dR \lesssim 1$$

whereas the third is the same as the second in the range $0 \leqslant R \leqslant \eta^{-\frac{1}{2}}$, whereas in the range $R \geqslant \eta^{-\frac{1}{2}}$ we need to integrate by parts; schematically, this amounts to

$$\left| \int_{\eta^{-\frac{1}{2}}}^{\infty} \langle R\rangle^{-\frac{1}{2}} \eta^{-\frac{1}{4}} \partial_R e^{iR\eta^{\frac{1}{2}}} \, dR \right| \lesssim 1.$$

The full details are essentially the same as in the previous integration by parts step and we skip them.

Next, we extract the δ measure that sits on the diagonal of the kernel K from the representation formula (5.3), see also (5.4). To do so, we can restrict ξ, η to a compact subset of $(0, \infty)$. This is convenient, as we then have the following asymptotics of $\phi(R, \xi)$ for $R\xi^{\frac{1}{2}} \gg 1$:

$$\phi(R,\xi) = \mathrm{Re}\left[a(\xi)\xi^{-\frac{1}{4}} e^{iR\xi^{\frac{1}{2}}}\left(1 + \frac{15i}{8R\xi^{\frac{1}{2}}}\right)\right] + O\big(R^{-2}\big),$$

$$(R\partial_R - 2\xi\partial_\xi)\phi(R,\xi) = -2\,\mathrm{Re}\left[\xi\partial_\xi\big(a(\xi)\xi^{-\frac{1}{4}}\big) e^{iR\xi^{\frac{1}{2}}}\left(1 + \frac{15i}{8R\xi^{\frac{1}{2}}}\right)\right] + O\big(R^{-2}\big)$$

where the $O(\cdot)$ terms depend on the choice of the compact subset. The R^{-2} terms are integrable so they contribute a bounded kernel to the inner product in (5.3). The same applies to the contribution of a bounded R region. Using the above expansions, we conclude that the δ-measure contribution of the inner product in (5.3) can only come from one of the following integrals:

$$-\int_0^\infty\int_0^\infty f(\xi)\chi(R)\,\mathrm{Re}\left[\xi\partial_\xi\big(a(\xi)\xi^{-\frac{1}{4}}\big)a(\eta)\eta^{-\frac{1}{4}} e^{iR(\xi^{\frac{1}{2}}+\eta^{\frac{1}{2}})} \times\left(1+\frac{15i}{8R\xi^{\frac{1}{2}}}\right)\left(1+\frac{15i}{8R\eta^{\frac{1}{2}}}\right)\right]\rho(\xi)\,d\xi\,dR \tag{5.18}$$

$$-\frac{1}{2}\int_0^\infty\int_0^\infty f(\xi)\chi(R)\,\xi\partial_\xi\big(a(\xi)\xi^{-\frac{1}{4}}\big)\bar{a}(\eta)\eta^{-\frac{1}{4}} e^{iR(\xi^{\frac{1}{2}}-\eta^{\frac{1}{2}})} \times\left(1+\frac{15i}{8R\xi^{\frac{1}{2}}}\right)\left(1-\frac{15i}{8R\eta^{\frac{1}{2}}}\right)\rho(\xi)\,d\xi\,dR \tag{5.19}$$

$$-\frac{1}{2}\int_0^\infty\int_0^\infty f(\xi)\chi(R)\,\xi\partial_\xi\big(\bar{a}(\xi)\xi^{-\frac{1}{4}}\big)a(\eta)\eta^{-\frac{1}{4}} e^{-iR(\xi^{\frac{1}{2}}-\eta^{\frac{1}{2}})} \times\left(1-\frac{15i}{8R\xi^{\frac{1}{2}}}\right)\left(1+\frac{15i}{8R\eta^{\frac{1}{2}}}\right)\rho(\xi)\,d\xi\,dR \tag{5.20}$$

where χ is a smooth cutoff function which equals 0 near $R = 0$ and 1 near $R = \infty$. In all of the above integrals we can argue as in the proof of the classical Fourier inversion formula to change the order of integration. Integrating by parts in the first integral (5.18) reveals that it cannot contribute a δ-measure. Discarding the R^{-2} terms from (5.19) and (5.20) reduces us further to the expressions

$$-\int_0^\infty\int_0^\infty f(\xi)\chi(R)\operatorname{Re}\bigl[\xi\partial_\xi\bigl(a(\xi)\xi^{-\frac14}\bigr)\bar a(\eta)\eta^{-\frac14}e^{iR(\xi^{\frac12}-\eta^{\frac12})}\bigr]\rho(\xi)\,d\xi\,dR \tag{5.21}$$

$$+\frac{15}{8}\int_0^\infty\int_0^\infty f(\xi)\chi(R)\operatorname{Im}\bigl[\xi\partial_\xi\bigl(a(\xi)\xi^{-\frac14}\bigr)\bar a(\eta)\eta^{-\frac14}e^{iR(\xi^{\frac12}-\eta^{\frac12})}\bigr]R^{-1}\bigl(\xi^{-\frac12}-\eta^{-\frac12}\bigr)\rho(\xi)\,d\xi\,dR. \tag{5.22}$$

The second integral (5.22) has both an R^{-1} and a $(\xi^{-\frac12}-\eta^{-\frac12})$ factor so its contribution to K is bounded. The first integral (5.21) contributes both a Hilbert transform type kernel as well as a δ-measure to K. By inspection, the δ contribution is

$$\begin{aligned}
&-\frac12\int_{-\infty}^\infty \operatorname{Re}\bigl[\xi\partial_\xi\bigl(a(\xi)\xi^{-\frac14}\bigr)\bar a(\eta)\eta^{-\frac14}e^{iR(\xi^{\frac12}-\eta^{\frac12})}\bigr]\rho(\xi)\,dR\\
&\quad=-\pi\operatorname{Re}\bigl[\xi\partial_\xi\bigl(a(\xi)\xi^{-\frac14}\bigr)\bar a(\eta)\eta^{-\frac14}\bigr]\rho(\xi)\delta\bigl(\xi^{\frac12}-\eta^{\frac12}\bigr)\\
&\quad=-2\pi\xi^{\frac12}\rho(\xi)\operatorname{Re}\bigl[\xi\partial_\xi\bigl(a(\xi)\xi^{-\frac14}\bigr)\bar a(\xi)\xi^{-\frac14}\bigr]\delta(\xi-\eta)\\
&\quad=-2\pi\xi^{\frac12}\rho(\xi)\operatorname{Re}\biggl[-\frac14\xi^{-\frac12}\bigl|a(\xi)\bigr|^2+\xi^{\frac12}a(\xi)\bar a'(\xi)\biggr]\delta(\xi-\eta)\\
&\quad=\biggl[\frac12+\frac{\xi\rho'(\xi)}{\rho(\xi)}\biggr]\delta(\xi-\eta)
\end{aligned}$$

where we used that $\rho(\xi)^{-1}=\pi|a|^2$ in the final step. Combining this with the δ-measure in (5.3) yields (5.4).

(b) Arguing as in part (a) we have

$$K_{\mathrm e}(\eta)=\frac{F(0,\eta)}{\eta}.$$

For F we use the representation in (5.11) with ξ replaced by 0 and $\phi(\cdot,\xi)$ replaced by ϕ_0. The conclusion easily follows from pointwise bounds on $\phi(\cdot,\eta)$ and its derivatives. $\square$

Next we consider the L^2 mapping properties for $\mathcal K$. We introduce the weighted L^2 spaces $L^{2,\alpha}_\rho$ of functions on $\operatorname{spec}(\mathcal L)$ with norm

$$\|f\|^2_{L^{2,\alpha}_\rho}:=\bigl|f(0)\bigr|^2+\int_0^\infty\bigl|f(\xi)\bigr|^2\langle\xi\rangle^{2\alpha}\rho(\xi)\,d\xi. \tag{5.23}$$

Then we have

Proposition 5.2.

(a) *The operators $\mathcal{K}_0$, $\mathcal{K}$ map*

$$\mathcal{K}_0 : L^{2,\alpha}_\rho \to L^{2,\alpha+1/2}_\rho, \qquad \mathcal{K} : L^{2,\alpha}_\rho \to L^{2,\alpha}_\rho.$$

(b) *In addition, we have the commutator bound*

$$[\mathcal{K}, \xi\partial_\xi] : L^{2,\alpha}_\rho \to L^{2,\alpha}_\rho$$

with $\xi\partial_\xi$ acting only on the continuous spectrum. Both statements hold for all $\alpha \in \mathbb{R}$.

Proof. We commence with the $\mathcal{K}_0$ part. (a) The first property is equivalent to showing that the kernel

$$\rho^{\frac{1}{2}}(\eta)\langle\eta\rangle^{\alpha+1/2} K_0(\eta,\xi)\langle\xi\rangle^{-\alpha}\rho^{-\frac{1}{2}}(\xi) : L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+).$$

With the notation of the previous theorem, the kernel on the left-hand side is

$$\tilde{K}_0(\eta,\xi) := \langle\eta\rangle^{\alpha+1/2}\langle\xi\rangle^{-\alpha}\frac{\sqrt{\rho(\xi)\rho(\eta)}}{\xi-\eta}F(\xi,\eta).$$

We first separate the diagonal and off-diagonal behavior of $\tilde{K}_0$, considering several cases.

Case 1. $(\xi,\eta) \in Q := [0,4] \times [0,4]$. We cover the unit interval with dyadic subintervals $I_j = [2^{j-1}, 2^{j+1}]$. We cover the diagonal with the union of squares

$$A = \bigcup_{j=-\infty}^{2} I_j \times I_j$$

and divide the kernel $\tilde{K}_0$ into

$$1_Q\tilde{K}_0 = 1_{A\cap Q}\tilde{K}_0 + 1_{Q\setminus A}\tilde{K}_0.$$

Case 1(a). Here we show that the diagonal part $1_{A\cap Q}\tilde{K}_0$ of $\tilde{K}_0$ maps L^2 to L^2. By orthogonality it suffices to restrict ourselves to a single square $I_j \times I_j$. We recall the $T1$ theorem for Calderon–Zygmund operators, see [16, p. 293]: suppose the kernel $K(\eta,\xi)$ on $\mathbb{R}^2$ defines an operator $T : \mathcal{S} \to \mathcal{S}'$ and has the following pointwise properties with some $\gamma \in (0,1]$ and a constant C_0:

(i) $|K(\eta,\xi)| \leqslant C_0|\xi-\eta|^{-1}$,
(ii) $|K(\eta,\xi) - K(\eta',\xi)| \leqslant C_0|\eta-\eta'|^\gamma|\xi-\eta|^{-1-\gamma}$ for all $|\eta-\eta'| < |\xi-\eta|/2$,
(iii) $|K(\eta,\xi) - K(\eta,\xi')| \leqslant C_0|\xi-\xi'|^\gamma|\xi-\eta|^{-1-\gamma}$ for all $|\xi-\xi'| < |\xi-\eta|/2$.

If in addition T has the restricted L^2 boundedness property, i.e., for all $r > 0$ and $\xi_0, \eta_0 \in \mathbb{R}$, $\|T(\omega^{r,\xi_0})\|_2 \leqslant C_0 r^{\frac{1}{2}}$ and $\|T^*(\omega^{r,\eta_0})\|_2 \leqslant C_0 r^{\frac{1}{2}}$ where $\omega^{r,\xi_0}(\xi) = \omega((\xi-\xi_0)/r)$ with a fixed bump-function ω, then T and T^* are $L^2(\mathbb{R})$ bounded with an operator norm that only depends on C_0.

Within the square $I_j \times I_j$, Theorem 5.1 shows that the kernel of $\tilde{K}_0$ satisfies these properties with $\gamma = 1$, and is thus bounded on L^2.

Case 1(b). Consider now the off-diagonal part $1_{Q\setminus A}\tilde{K}_0$. In this region, by Theorem 5.1,

$$\left|\tilde{K}_0(\eta,\xi)\right| \lesssim (\xi\eta)^{-\frac{1}{4}}$$

which is a Hilbert–Schmidt kernel on Q and thus L^2 bounded.

Case 2. $(\xi,\eta) \in Q^c$. We cover the diagonal with the union of squares

$$B = \bigcup_{j=1}^{\infty} I_j \times I_j$$

and divide the kernel $\tilde{K}_0$ into

$$1_{Q^c}\tilde{K}_0 = 1_{B\cap Q^c}\tilde{K}_0 + 1_{Q^c\setminus B}\tilde{K}_0.$$

Case 2(a). Here we consider the estimate on B. As in case 1(a) above, we use Calderon–Zygmund theory. Evidently, $|\tilde{K}_0(\eta,\xi)| \lesssim |\xi-\eta|^{-1}$ on B by Theorem 5.1. To check (ii) and (iii), we differentiate $\tilde{K}_0$. It will suffice to consider the case where the ∂_ξ derivative falls on $F(\xi,\eta)$. We distinguish two cases: if $|\xi^{\frac{1}{2}} - \eta^{\frac{1}{2}}| \leqslant 1$, then $|\xi-\eta| \lesssim \xi^{\frac{1}{2}}$ which implies that

$$\frac{\xi^{-\frac{1}{2}}|\xi-\xi'|}{|\xi-\eta|} \lesssim \frac{|\xi-\xi'|^{\frac{1}{2}}}{|\xi-\eta|^{\frac{3}{2}}} \quad \forall|\xi-\xi'| < |\xi-\eta|/2;$$

if, on the other hand, $|\xi^{\frac{1}{2}} - \eta^{\frac{1}{2}}| > 1$, then

$$\frac{\xi^{-\frac{1}{2}}|\xi-\xi'|}{|\xi-\eta||\xi^{\frac{1}{2}}-\eta^{\frac{1}{2}}|} \lesssim \frac{|\xi-\xi'|}{|\xi-\eta|^2} \quad \forall|\xi-\xi'| < |\xi-\eta|/2$$

which proves property (iii) on B with $\gamma = \frac{1}{2}$, and by symmetry also (ii). The restricted L^2 property follows form the cancellation in the kernel and the previous bounds on the kernel. Hence, $\tilde{K}_0$ is L^2 bounded on B.

Case 2(b). Finally, in the exterior region $Q^c \setminus B$ we have the bound, with arbitrarily large N,

$$\left|\tilde{K}_0(\eta,\xi)\right| \lesssim (1+\xi)^{-N}(1+\eta)^{-N}$$

which is L^2 bounded by Schur's lemma.

This concludes the proof of the first mapping property in part (a). The second one follows in a straightforward manner since K_{e} is rapidly decaying at ∞.

(b) A direct computation shows that the kernel K_0^{com} of the commutator $[\xi\partial_\xi, K_0]$ is given by

$$K_0^{\mathrm{com}}(\eta,\xi) = (\eta\partial_\eta + \xi\partial_\xi)K_0(\eta,\xi) + K_0(\eta,\xi) = \frac{\rho(\xi)}{\xi-\eta}F^{\mathrm{com}}(\xi,\eta)$$

interpreted in the principal value sense and with F^{com} given by

$$F^{\mathrm{com}}(\xi,\eta) = \frac{\xi\rho'(\xi)}{\rho(\xi)}F(\xi,\eta) + (\xi\partial_\xi + \eta\partial_\eta)F(\xi,\eta).$$

By Theorem 5.1 this satisfies the same pointwise off-diagonal bounds as F. Near the diagonal the bounds for F^{com} and its derivatives are worse than those for F by a factor of $(1+\xi)^{\frac{1}{2}}$. Then the proof of the L^2 commutator bound for K_0 is similar to the argument in part (a).

The remaining part of the commutator $[\mathcal{K}, \xi\partial_\xi]$ involves:

(i) The commutator of the diagonal part of $\mathcal{K}_{\mathrm{cc}}$ with $\xi\partial_\xi$. This is the multiplication operator by

$$\xi\partial_\xi\frac{\xi\rho'(\xi)}{\rho(\xi)}$$

which is bounded since ρ has symbol like behavior both at 0 and at ∞.

(ii) The operator $\xi\partial_\xi\mathcal{K}_{\mathrm{ce}}$ which is given by the bounded rapidly decreasing function $\xi\partial_\xi K_{\mathrm{e}}(\xi)$.

(iii) The operator $\mathcal{K}_{\mathrm{ec}}\xi\partial_\xi$ given by

$$\mathcal{K}_{\mathrm{ec}}\xi\partial_\xi f = \int_0^\infty K_{\mathrm{e}}(\xi)\xi\partial_\xi f(\xi)\,d\xi = -\int_0^\infty f(\xi)\partial_\xi\big(\xi K_{\mathrm{e}}(\xi)\big)\,d\xi$$

which is also bounded due to the properties of K_{e}. $\square$

6. The second order transport equation

This section is devoted to the study of the linear problem (2.2) which we restate here in the form

$$\big(-\partial_t^2 + \partial_r^2 + r^{-1}\partial_r + 2r^{-2}\big(1-3Q(R)^2\big)\big)\varepsilon = f - 12\omega^2\frac{R^2(1-R^2)}{(1+R^2)^3}\varepsilon. \tag{6.1}$$

We recall that the second term on the right-hand side here arises due to fact that its decay is of the same nature (namely ω^2) as that of other error terms which we will encounter in the parametrix construction of this section. By doing this, the remaining terms in the nonlinearity $\mathcal{N}$ in (2.2) decay more rapidly at infinity. Our main result asserts that

Proposition 6.1. *The backward solution ε for* (6.1) *satisfies the bound*

$$\|\varepsilon\|_{H_N^1} \lesssim \frac{1}{N}\|f\|_{L_N^2} \tag{6.2}$$

for all large enough N.

Proof. We work in the coordinates (R,τ) given by

$$R = r\lambda(t), \qquad \tau = \int_t^1 \lambda(s)\,ds = (\beta+1)^{-1}|\log t|^{\beta+1}$$

for any $0 < t < 1$. For future reference, we note that

$$t\lambda(t) = \big((\beta+1)\tau\big)^{\frac{\beta}{\beta+1}}, \qquad \lambda(\tau) = \big((\beta+1)\tau\big)^{\frac{\beta}{\beta+1}} e^{((\beta+1)\tau)^{\frac{1}{\beta+1}}}.$$

We introduce the auxiliary weight function $\omega(\tau)$

$$\omega(\tau) := \lambda^{-1}\lambda_\tau(\tau) = \frac{\beta}{\beta+1}\tau^{-1} + \big((\beta+1)\tau\big)^{-\frac{\beta}{\beta+1}}$$

and note that

$$(t\lambda)^{-1} = \omega(\tau) - \frac{\beta}{\beta+1}\tau^{-1}. \tag{6.3}$$

Then

$$\partial_t = \frac{\partial\tau}{\partial t}(\partial_\tau + R_\tau\partial_R) = -\lambda(\tau)(\partial_\tau + \omega R\partial_R),$$
$$\partial_t^2 = \lambda^2(\tau)\big[(\partial_\tau + \omega R\partial_R)^2 + \omega(\partial_\tau + \omega R\partial_R)\big]$$

therefore Eq. (6.1) becomes

$$\left[-(\partial_\tau + \omega R\partial_R)^2 - \omega(\partial_\tau + \omega R\partial_R) + \partial_R^2 + \frac{1}{R}\partial_R + \frac{2}{R^2}\big(1 - 3Q(R)^2\big)\right]\varepsilon$$
$$= \lambda^{-2} f - 12\omega^2 \frac{R^2(1-R^2)}{(1+R^2)^3}\varepsilon.$$

At this point it is convenient to switch to the notations

$$\tilde\varepsilon(\tau,R) = R^{\frac{1}{2}}\varepsilon(\tau,R), \qquad \tilde f(\tau,R) := R^{\frac{1}{2}}\lambda^{-2} f(\tau,R). \tag{6.4}$$

Since

$$R^{\frac{1}{2}}(\partial_\tau + \omega R\partial_R)R^{-\frac{1}{2}} = \partial_\tau + \omega R\partial_R - \omega/2,$$

one concludes from conjugating the previous PDE by $R^{\frac{1}{2}}$ that

$$\left[-(\partial_\tau + \omega R\partial_R)^2 + \frac{1}{2}\dot\omega + \frac{1}{4}\omega^2 - \mathcal{L}\right]\tilde\varepsilon = \tilde f - 12\omega^2 \frac{R^2(1-R^2)}{(1+R^2)^3}\tilde\varepsilon \tag{6.5}$$

where $\dot\omega := \partial_\tau \omega$ and

$$\mathcal{L} := -\partial_R^2 + \frac{15}{4R^2} - \frac{24}{(1+R^2)^2}.$$

Written in terms of $(\tilde\varepsilon, \tilde f)$ the estimate (6.2) takes the form[6]

$$\|\tilde\varepsilon\|_{\tilde H^1_N} \lesssim \frac{1}{N}\|\tilde f\|_{\tilde L^2_N} \tag{6.6}$$

where

$$\|\tilde\varepsilon\|_{\tilde H^1_N} = \sup_{\tau>\tau_0} \tau^{N-1-\frac{\beta}{\beta+1}}\big\|\tilde\varepsilon(\tau)\big\|_{L^2} + \tau^{N-1}\big(\big\|\mathcal{L}^{\frac12}\tilde\varepsilon(\tau)\big\|_{L^2} + \big\|(\partial_\tau + \omega R\partial_R)\tilde\varepsilon(\tau)\big\|_{L^2}\big),$$

respectively

$$\|\tilde f\|_{\tilde L^2_N} = \sup_{\tau>\tau_0} \tau^N \big\|\tilde f(\tau)\big\|_{L^2}.$$

In order to take advantage of the spectral properties of the operator $\mathcal{L}$ we conjugate Eq. (6.5) by the Fourier transform $\mathcal{F}$ adapted to $\mathcal{L}$. The transference identity is

$$\mathcal{F} R\partial_R \mathcal{F}^{-1} = -2\xi\partial_\xi + \mathcal{K}$$

where

$$\mathcal{K} = \begin{bmatrix} -\frac12 & \mathcal{K}_{\mathrm{ec}} \\ \mathcal{K}_{\mathrm{ce}} & \mathcal{K}_{\mathrm{cc}} \end{bmatrix} = -\frac12 \,\mathrm{Id} + \begin{bmatrix} 0 & \mathcal{K}_{\mathrm{ec}} \\ \mathcal{K}_{\mathrm{ce}} & -(1+\eta\rho'(\eta)/\rho(\eta))\delta(\xi-\eta) + \mathcal{K}_0 \end{bmatrix}$$

and

$$\mathcal{K}_{\mathrm{ec}} f = -\int_0^\infty f(\xi) K_{\mathrm{e}}(\xi)\rho(\xi)\,d\xi, \qquad \mathcal{K}_{\mathrm{ce}} = K_{\mathrm{e}},$$

$$K_{\mathrm{e}}(\xi) = \big\langle R\phi_0'(R), \phi(R,\xi)\big\rangle.$$

We write

$$-\mathcal{F} R\partial_R \mathcal{F}^{-1} = \frac12 \,\mathrm{Id} + \mathcal{K}_{\mathrm{d}} + \mathcal{K}_{\mathrm{nd}}$$

where

$$\mathcal{K}_{\mathrm{d}} = \begin{bmatrix} 0 & 0 \\ 0 & 2\xi\partial_\xi + (1+\xi\rho'(\xi)/\rho(\xi)) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \mathcal{D}_0 \end{bmatrix},$$

$$\mathcal{K}_{\mathrm{nd}} = -\begin{bmatrix} 0 & \mathcal{K}_{\mathrm{ec}} \\ \mathcal{K}_{\mathrm{ce}} & \mathcal{K}_0 \end{bmatrix}.$$

[6] Here we slightly abuse notations since the N's in (6.2) and (6.6) do not coincide, instead they are linearly related.

Then

$$\mathcal{F}(\partial_\tau + \omega R\partial_R)\mathcal{F}^{-1} = D_\tau + \frac{\omega}{2} - \omega\mathcal{K}_{\mathrm{nd}}, \quad D_\tau = \partial_\tau - \omega(1+\mathcal{K}_{\mathrm{d}}),$$

therefore

$$\mathcal{F}(\partial_\tau + \omega R\partial_R)^2\mathcal{F}^{-1} = \left(D_\tau + \frac{\omega}{2}\right)^2 - 2\omega\mathcal{K}_{\mathrm{nd}} D_\tau + \omega^2\big([\mathcal{K}_{\mathrm{nd}}, \mathcal{K}_{\mathrm{d}}] + \mathcal{K}_{\mathrm{nd}}^2 - \mathcal{K}_{\mathrm{nd}}\big) - \dot{\omega}\mathcal{K}_{\mathrm{nd}}.$$

Next we consider the Fourier transform of the last term in (6.5), which we express in the form

$$\mathcal{F}\left(-12\frac{R^2(1-R^2)}{(1+R^2)^3}\tilde{\varepsilon}\right) = \mathcal{J}\mathcal{F}\tilde{\varepsilon}, \qquad \mathcal{J} = \begin{bmatrix} \mathcal{J}_{\mathrm{ee}} & \mathcal{J}_{\mathrm{ec}} \\ \mathcal{F}_{\mathrm{ce}} & \mathcal{J}_{\mathrm{cc}} \end{bmatrix}.$$

We note that

$$\mathcal{J}_{\mathrm{ee}} = \|\phi_0\|_{L^2}^{-2}\left\langle\phi_0, \frac{-12R^2(1-R^2)}{(1+R^2)^3}\phi_0\right\rangle = \left(\frac{1}{6}\right)^{-1}\frac{1}{10} = \frac{3}{5}$$

while

$$\mathcal{J}_{\mathrm{ce}} = J_{\mathrm{e}}(\xi) = \left\langle\phi(R,\xi), \frac{-12R^2(1-R^2)}{(1+R^2)^3}\phi_0\right\rangle$$

and

$$\mathcal{J}_{\mathrm{ec}} x = \int_0^\infty \rho(\xi) J_{\mathrm{e}}(\xi) x(\xi)\, d\xi.$$

We remark that the kernel J_{e} is bounded and rapidly decreasing at infinity. Finally,

$$\mathcal{J}_{\mathrm{cc}} x(\xi) = \int_0^\infty \rho(\xi) J_{\mathrm{cc}}(\xi,\eta) x(\eta)\, d\eta$$

with

$$J_{\mathrm{cc}}(\xi,\eta) = \int_0^\infty -12\frac{R^2(1-R^2)}{(1+R^2)^3}\rho(\eta)\phi(R,\eta)\phi(R,\xi)\, dR.$$

This is bounded and has the off-diagonal decay property

$$J_{\mathrm{cc}}(\xi,\eta)| \lesssim (1+\xi)^{-\frac{1}{2}}\big(1 + \big|\xi^{\frac{1}{2}} - \eta^{\frac{1}{2}}\big|\big)^{-N}. \tag{6.7}$$

Taking into account all the notations above, Eq. (6.5) becomes

$$\begin{aligned}\big[-D_\tau^2 - \omega D_\tau - \xi\big]\mathcal{F}\tilde{\varepsilon} &= \mathcal{F}\tilde{f} - 2\omega\mathcal{K}_{\mathrm{nd}} D_\tau \mathcal{F}\tilde{\varepsilon} \\ &\quad + \omega^2\big([\mathcal{K}_{\mathrm{nd}}, \mathcal{K}_{\mathrm{d}}] + \mathcal{K}_{\mathrm{nd}}^2 - \mathcal{K}_{\mathrm{nd}} + \mathcal{J}\big)\mathcal{F}\tilde{\varepsilon} - \dot{\omega}\mathcal{K}_{\mathrm{nd}}\mathcal{F}\tilde{\varepsilon}.\end{aligned}$$

Next, write $\mathcal{F}\tilde{\varepsilon} = \left[\begin{smallmatrix} x_0 \\ x \end{smallmatrix}\right]$ and $\mathcal{F}\tilde{f} = \left[\begin{smallmatrix} g_0 \\ g \end{smallmatrix}\right]$, or equivalently,

$$\tilde{\varepsilon}(\tau, R) = x_0(\tau)\phi_0(R)\|\phi_0\|_2^{-2} + \int_0^\infty x(\tau, \xi)\phi(R, \xi)\rho(\xi)\, d\xi =: \tilde{\varepsilon}_0 + \tilde{\varepsilon}_{\mathrm{c}}$$

where $\tilde{\varepsilon}_0 \perp \tilde{\varepsilon}_{\mathrm{c}}$ for all $\tau \geqslant 0$ (recall $\phi_0(R) = R^{\frac{5}{2}}(1+R^2)^{-2}$ and $\mathcal{L}\phi_0 = 0$). To write the system for $\left[\begin{smallmatrix} x_0 \\ x \end{smallmatrix}\right]$ we compute

$$\begin{aligned}
\mathcal{K}_{\mathrm{nd}}^2 &= \begin{bmatrix} 0 & \mathcal{K}_{\mathrm{ec}} \\ \mathcal{K}_{\mathrm{ce}} & \mathcal{K}_0 \end{bmatrix}^2 = \begin{bmatrix} \mathcal{K}_{\mathrm{ec}}\mathcal{K}_{\mathrm{ce}} & \mathcal{K}_{\mathrm{ec}}\mathcal{K}_0 \\ \mathcal{K}_0\mathcal{K}_{\mathrm{ce}} & \mathcal{K}_{\mathrm{ce}}\mathcal{K}_{\mathrm{ec}} + \mathcal{K}_0^2 \end{bmatrix}, \\
\mathcal{K}_{\mathrm{nd}}\mathcal{K}_{\mathrm{d}} &= -\begin{bmatrix} 0 & \mathcal{K}_{\mathrm{ec}} \\ \mathcal{K}_{\mathrm{ce}} & \mathcal{K}_0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & \mathcal{D}_0 \end{bmatrix} = \begin{bmatrix} 0 & -\mathcal{K}_{\mathrm{ec}}\mathcal{D}_0 \\ 0 & -\mathcal{K}_0\mathcal{D}_0 \end{bmatrix}, \\
\mathcal{K}_{\mathrm{d}}\mathcal{K}_{\mathrm{nd}} &= -\begin{bmatrix} 0 & 0 \\ 0 & \mathcal{D}_0 \end{bmatrix}\begin{bmatrix} 0 & \mathcal{K}_{\mathrm{ec}} \\ \mathcal{K}_{\mathrm{ce}} & \mathcal{K}_0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -\mathcal{D}_0\mathcal{K}_{\mathrm{ce}} & -\mathcal{D}_0\mathcal{K}_0 \end{bmatrix}.
\end{aligned}$$

We also note that

$$-K_{\mathrm{ec}}K_{\mathrm{ce}} = \int_0^\infty \rho(\xi)\big|K_{\mathrm{e}}(\xi)\big|^2\, d\xi = \frac{\|R\partial_R\phi_0\|_{L^2}^2}{\|\phi_0\|_{L^2}^2} - \frac{\langle R\partial_R\phi_0, \phi_0\rangle^2}{\|\phi_0\|_{L^2}^4} = 6\frac{17}{120} - \frac{1}{4} = \frac{3}{5}.$$

Then we seek to write the equations for x_0 and x in the form of a diagonal system with perturbative coupling,

$$P\begin{bmatrix} x_0 \\ x \end{bmatrix} = \begin{bmatrix} g_0 \\ g \end{bmatrix} + Q\begin{bmatrix} x_0 \\ x \end{bmatrix} \tag{6.8}$$

where

$$P = \begin{bmatrix} P_{\mathrm{e}} & 0 \\ 0 & P_{\mathrm{c}} \end{bmatrix}, \qquad Q = \begin{bmatrix} 0 & Q_{\mathrm{ec}} \\ Q_{\mathrm{ce}} & Q_{\mathrm{cc}} \end{bmatrix}$$

with the principal part given by

$$P_{\mathrm{e}} = -\partial_\tau(\partial_\tau - \omega),$$

respectively

$$P_{\mathrm{c}} = -D_\tau^2 - \omega D_\tau - \xi$$

and the coupling terms of the form

$$\mathcal{Q}_{\mathrm{ec}} x = \omega^2 \mathcal{R}_{\mathrm{ec}} x - 2\omega \mathcal{K}_{\mathrm{ec}} D_\tau x$$

with

$$\mathcal{R}_{\mathrm{ec}} = \big(\omega^{-2}\dot{\omega} - 1\big)\mathcal{K}_{\mathrm{ec}} + \mathcal{K}_{\mathrm{ec}}\mathcal{K}_0 + \mathcal{J}_{\mathrm{ec}}$$

while

$$\mathcal{Q}_{\mathrm{ce}} x_0 = \omega^2 \mathcal{R}_{\mathrm{ce}} x_0 - 2\omega \mathcal{K}_{\mathrm{ce}} \partial_\tau x_0, \qquad \mathcal{Q}_{\mathrm{cc}} x = \omega^2 \mathcal{R}_{\mathrm{cc}} x - 2\omega \mathcal{K}_0 D_\tau x$$

with

$$\mathcal{R}_{\mathrm{cc}} = [\mathcal{K}_0, \mathcal{B}_0] + \mathcal{K}_0^2 + \mathcal{K}_{\mathrm{ce}}\mathcal{K}_{\mathrm{ec}} + \mathcal{J}_{\mathrm{cc}},$$

respectively

$$\mathcal{R}_{\mathrm{ce}} = -\mathcal{B}_0\mathcal{K}_{\mathrm{ce}} - \mathcal{K}_0\mathcal{K}_{\mathrm{ce}} + \mathcal{J}_{\mathrm{ce}}.$$

Our main solvability result in Proposition 6.2 for Eq. (6.1) is restated in terms of the system (6.8) as follows:

Proposition 6.2. *For each with* (g_0, g) *which satisfy*

$$\big|g_0(\tau)\big| \leqslant \tau^{-N}, \qquad \big\|g(\tau,\cdot)\big\|_{L^2_\rho} \leqslant \tau^{-N},$$

there exists an unique solution (x, x_0) *for the system* (6.8) *decaying at infinity. This solution satisfies the bounds*

$$\big|x_0(\tau)\big| \lesssim \frac{1}{N}\tau^{-N+\frac{2\beta+1}{\beta+1}}, \qquad |\partial_\tau x_0| \lesssim \tau^{-N+\frac{\beta}{\beta+1}}, \tag{6.9}$$

respectively

$$\big\|x(\tau)\big\|_{L^2_\rho} \lesssim \frac{1}{N}\tau^{-N+\frac{2\beta+1}{\beta+1}}, \qquad \big\|\xi^{\frac{1}{2}} x(\tau)\big\|_{L^2_\rho} + \big\|D_\tau x(\tau)\big\|_{L^2_\rho} \lesssim \frac{1}{N}\tau^{-N+1}. \tag{6.10}$$

Proof. Our strategy is to solve first the simpler linear equations

$$-\partial_\tau(\partial_\tau - \omega)x_0 = g_0, \tag{6.11}$$

$$\big[-D_\tau^2 - \omega D_\tau - \xi\big]x = g. \tag{6.12}$$

Then we will show that the right-hand side in the system (6.8) is perturbative. We start with the linear operator governing x_0, and introduce the appropriate function spaces for x_0 and g_0:

$$\|g_0\|_{Y_0^N} = \sup_{\tau \geqslant \tau_0} \tau^N \left|g_0(\tau)\right|,$$

$$\|x_0\|_{X_0^N} = \sup_{\tau \geqslant \tau_0} \tau^{N-\frac{2\beta+1}{\beta+1}} \left|x_0(\tau)\right| + \tau^{N-\frac{\beta}{\beta+1}} \left|\partial_\tau x_0(\tau)\right|.$$

Lemma 6.3. *The backward solution operator $x_0 = T_{\mathrm{e}} g_0$ for* (6.11) *satisfies the estimate*

$$\|T_{\mathrm{e}} g_0\|_{X_0^N} \lesssim \left\|g_0(\sigma)\right\|_{Y_0^N} \tag{6.13}$$

for any $N \geqslant 2$.

Proof. A fundamental basis of solutions of $-\partial_\tau(\partial_\tau - \omega)$ is given by

$$a_+(\tau) = \lambda(\tau), \qquad a_-(\tau) = \lambda(\tau) \int_\tau^\infty \lambda^{-1}(\sigma)\, d\sigma = \omega^{-1}(\tau)\bigl(1 + O\bigl(\tau^{-\frac{1}{\beta+1}}\bigr)\bigr)$$

and has Wronskian $W(\tau) = \lambda(\tau)$. Then the backward fundamental solution is given by

$$U_0(\tau, \sigma) = W^{-1}(\sigma)\bigl(a_+(\tau)a_-(\sigma) - a_+(\sigma)a_-(\tau)\bigr) = \lambda(\tau) \int_\tau^\sigma \lambda^{-1}(s)\, ds. \tag{6.14}$$

A direct computation shows that U_0 satisfies the bounds

$$\left|U_0(\tau, \sigma)\right| \lesssim \frac{1}{\omega(\tau)}, \qquad \left|\partial_\tau U_0(\tau, \sigma)\right| \lesssim \tau^{-\frac{1}{\beta+1}} + \frac{\omega(\tau)}{\omega(\sigma)} \frac{\lambda(\tau)}{\lambda(\sigma)}.$$

The conclusion of the lemma follows. □

Next we bound the solution of Eq. (6.12), which is hyperbolic. One is tempted to define spaces X^N and Y^N in a manner which is similar to X_0^N and Y_0^N. This would work for the linear theory for (6.12), but would not be strong enough in order to treat the right-hand side in (6.8) in a perturbative manner. Instead we define some stronger spaces using the additional weight

$$m(\xi) = \xi^\nu + \xi^{-\nu}$$

where $\nu > 0$ is a fixed small parameter. We define the space $L_N^\infty L_\rho^2$ with norm

$$\|g\|_{L_N^\infty L_\rho^2} = \sup_{\tau > \tau_0} \tau^N \|g_1\|_{L_\rho^2}$$

and the dyadic L^2 space $l_N^\infty L_{\rho m}^2$ with norm

$$\|g\|_{l_N^\infty L_{\rho m}^2} = \sup_{\tau > \tau_0} \left\|\sigma^{N-\frac{\beta}{2(\beta+1)}} g(\sigma)\right\|_{L_{\rho m}^2([\tau, 2\tau]\times\mathbb{R})}.$$

Then we define the Y^N space as a sum of two spaces,

$$\|g\|_{Y^N} = \inf_{g=g_1+g_2} \|g_1\|_{L^\infty_N L^2_\rho} + \big\|g_2(\sigma)\big\|_{l^\infty_{N-\frac{1}{\beta+1}} L^2_{\rho m}}.$$

Similarly we introduce the X^N space with norm

$$\|x\|_{X^N} = \|x\|_{L^\infty_{N-1-\frac{\beta}{\beta+1}} L^2_\rho} + \big\|\big(\xi^{\frac{1}{2}}x, D_\tau x\big)\big\|_{L^\infty_{N-1} L^2_\rho \cap l^\infty_{N-1} L^2_{\rho/m}}.$$

Then our solvability result for (6.12) is as follows:

Lemma 6.4. *The backward solution operator $x = T_{\mathrm{c}} g$ for Eq.* (6.12) *satisfies*

$$\|T_{\mathrm{c}} g\|_{X^N} \lesssim \|g\|_{Y^N}. \tag{6.15}$$

In addition we have the smallness relation

$$\|T_{\mathrm{c}} g\|_{X^N} \lesssim \frac{1}{\sqrt{N}} \|g\|_{L^\infty_N L^2_\rho} \tag{6.16}$$

for large N.

Proof. Eq. (6.12) is equivalent to

$$\big[-(\partial_\tau - 2\omega\xi\partial_\xi)^2 + \omega(\partial_\tau - 2\omega\xi\partial_\xi) - \xi\big]\lambda^{-2}\rho^{\frac{1}{2}}(\xi)x = \lambda^{-2}\rho^{\frac{1}{2}}(\xi)g. \tag{6.17}$$

We substitute the functions (x, g) by (y, h) where $y = \rho^{\frac{1}{2}}x$ and $h = \rho^{\frac{1}{2}}g$. This has the effect of removing the weight ρ from the estimates. The functions (y, h) solve

$$\big[-(\partial_\tau - 2\omega\xi\partial_\xi)^2 + \omega(\partial_\tau - 2\omega\xi\partial_\xi) - \xi\big]\lambda^{-2}y = \lambda^{-2}h. \tag{6.18}$$

The characteristics of the homogeneous operator on the left are $(\tau, \lambda^{-2}(\tau)\xi_0)$ which means that

$$(\partial_\tau - 2\omega\xi\partial_\xi)f(\tau,\xi) = \frac{d}{d\tau}f\big(\tau,\xi(\tau)\big), \quad \xi(\tau) = \lambda^{-2}(\tau)\xi_0.$$

Hence, we are reduced to solving the ODE

$$\big[-\partial_\tau^2 + \omega(\tau)\partial_\tau - \lambda^{-2}(\tau)\xi_0\big]\lambda^{-2}y\big(\tau,\xi(\tau)\big) = \lambda^{-2}h\big(\tau,\xi(\tau)\big) \tag{6.19}$$

with $\xi_0 > 0$ fixed. The homogeneous equation has exact solutions

$$\big[-\partial_\tau^2 + \omega(\tau)\partial_\tau - \lambda^{-2}(\tau)\xi_0\big]e^{\pm i\xi_0^{\frac{1}{2}}\int_\tau^\infty \lambda^{-1}(\sigma)\,d\sigma} = 0.$$

This is no surprise since Eq. (6.12) is equivalent to the constant coefficient wave equation in the t, r coordinates.

Since the Wronskian

$$W\left(e^{i\xi_0^{\frac{1}{2}}\int_\tau^\infty \lambda^{-1}(\sigma)\,d\sigma}, e^{-i\xi_0^{\frac{1}{2}}\int_\tau^\infty \lambda^{-1}(\sigma)\,d\sigma}\right) = 2i\xi_0^{\frac{1}{2}}\lambda^{-1}(\tau),$$

it follows that the backward solution to (6.18) has the form

$$y(\tau,\xi_0) = \xi_0^{-\frac{1}{2}}\int_\tau^\infty \frac{\lambda^2(\tau)}{\lambda(\sigma)}\sin\left(\xi_0^{\frac{1}{2}}\int_\tau^\sigma \lambda^{-1}(u)\,du\right)h\big(\sigma,\xi(\sigma)\big)\,d\sigma.$$

Define the forward Green function

$$U(\tau,\sigma;\xi) := \xi^{-\frac{1}{2}}\frac{\lambda(\tau)}{\lambda(\sigma)}\sin\left(\xi^{\frac{1}{2}}\lambda(\tau)\int_\tau^\sigma \lambda^{-1}(u)\,du\right).$$

Since $\xi_0 = \lambda^2(\tau)\xi$, $\xi(\sigma) = \xi\lambda^2(\tau)\lambda^{-2}(\sigma)$, we can write

$$y(\tau,\xi) = \int_\tau^\infty U(\tau,\sigma;\xi)h\big(\sigma,\xi(\sigma)\big)\,d\sigma.$$

To estimate $D_\tau y$ it is also convenient to evaluate directly

$$D_\tau U(\tau,\sigma;\xi) = \frac{\lambda(\tau)}{\lambda(\sigma)}\cos\left(\xi^{\frac{1}{2}}\lambda(\tau)\int_\tau^\sigma \lambda^{-1}(u)\,du\right). \tag{6.20}$$

To estimate the solution y we either bound $|\sin(v)| \leqslant |v|$ or $|\sin(v)| \leqslant 1$. Using that

$$\lambda(\tau)\int_\tau^\infty \lambda^{-1}(u)\,du \lesssim \omega^{-1}(\tau)$$

one obtains

$$\big|U(\tau,\sigma;\xi)\big| \lesssim \omega^{-1}(\tau)\frac{\lambda(\tau)}{\lambda(\sigma)} \tag{6.21}$$

as well as

$$\xi^{\frac{1}{2}}\big|U(\tau,\sigma;\xi)\big| + \big|D_\tau U(\tau,\sigma;\xi)\big| \lesssim \frac{\lambda(\tau)}{\lambda(\sigma)}. \tag{6.22}$$

We denote

$$z(\tau,\xi) = \big(\omega(\tau)y(\tau,\xi), \xi^{\frac{1}{2}}y(\tau,\xi), D_\tau y(\tau,\xi)\big). \tag{6.23}$$

An immediate consequence of (6.21) and (6.22) is the estimate

$$\lambda^{-1}(\tau)\big|z\big(\tau,\xi(\tau)\big)\big| \lesssim \int_\tau^\infty \lambda^{-1}(\sigma)\big|h\big(\sigma,\xi(\sigma)\big)\big|\,d\sigma. \tag{6.24}$$

From this we need to conclude that the following four bounds hold:

$$\|z\|_{L^\infty_{N-1}L^2} \lesssim \frac{1}{N}\|h\|_{L^\infty_N L^2}, \tag{6.25}$$

$$\|z\|_{l^\infty_{N-1}L^2_{1/m}} \lesssim \frac{1}{\sqrt{N}}\|h\|_{L^\infty_N L^2}, \tag{6.26}$$

$$\|z\|_{L^\infty_{N-1}L^2} \lesssim \|h\|_{l^\infty_{N-\frac{1}{\beta+1}}L^2_m}, \tag{6.27}$$

respectively

$$\|z\|_{l^\infty_{N-1}L^2_{1/m}} \lesssim \|h\|_{l^\infty_{N-\frac{1}{\beta+1}}L^2_m}. \tag{6.28}$$

Taking L^2 norms in ξ on both sides of (6.24) we obtain

$$\big\|z(\tau)\big\|_{L^2} \lesssim \int_\tau^\infty \big\|h(\sigma)\big\|_{L^2}\,d\sigma$$

which leads directly to (6.25).

Adding flow invariant weights to the above bounds we get

$$\big\|z(\tau)\big\|_{L^2_{1/m}} \lesssim \int_\tau^\infty \left\|m^{-1}\left(\frac{\xi\lambda(\tau)}{\lambda(\sigma)}\right)h(\sigma)\right\|_{L^2} d\sigma$$

and by Cauchy–Schwarz

$$\big\|z(\tau)\big\|^2_{L^2_{1/m}} \lesssim \frac{1}{N}\int_\tau^\infty \frac{\sigma^N}{\tau^{N-1}}\left\|m^{-1}\left(\frac{\xi\lambda(\tau)}{\lambda(\sigma)}\right)h(\sigma)\right\|^2_{L^2} d\sigma.$$

Hence

$$\begin{aligned}
&\int_{\tau_1}^{2\tau_1} \tau^{2(N-1)-\frac{\beta}{\beta+1}}\big\|z(\tau)\big\|^2_{L^2_{1/m}}\,d\tau \\
&\quad\lesssim \frac{1}{N}\int_{\tau_1}^{2\tau_1}\int_\tau^\infty\int_0^\infty \tau^{2(N-1)-\frac{\beta}{\beta+1}}\frac{\sigma^N}{\tau^{N-1}} m^{-2}\left(\frac{\xi\lambda(\tau)}{\lambda(\sigma)}\right)\big|h(\sigma,\xi)\big|^2\,d\xi\,d\sigma\,d\tau
\end{aligned}$$

$$\lesssim \frac{1}{N}\int_{\tau_1}^{\infty}\sigma^N \left\|h(\sigma)\right\|_{L^2}^2 \sup_{\xi>0}\left(\int_{\tau_1}^{\min\{\sigma,2\tau_1\}}\tau^{N-1-\frac{\beta}{\beta+1}}m^{-2}\left(\frac{\xi\lambda(\tau)}{\lambda(\sigma)}\right)d\tau\right)d\sigma$$

$$\lesssim \frac{M}{N}\int_{\tau_1}^{\infty}\sigma^N \min\{\sigma,2\tau_1\}^{N-1}\left\|h(\sigma)\right\|_{L^2}^2\,d\sigma$$

$$\lesssim \frac{M}{N}\|h\|^2_{L_N^\infty L^2}$$

where

$$M=\sup_{\xi>0}\int_0^\infty \tau^{-\frac{\beta}{\beta+1}}m^{-2}\big(\xi\lambda(\tau)\big)\,d\tau \approx \sup_{\xi>0}\int_0^\infty m^{-2}(\xi\lambda)\frac{d\lambda}{\lambda}\approx 1.$$

This concludes the proof of (6.26).

We now turn our attention to (6.27), for which we need to take $h\in l^\infty_{N-\frac{1}{1+\beta}}L^2_m$. From (6.24) by Cauchy–Schwarz we obtain

$$\lambda^{-2}(\tau)\big|z\big(\tau,\xi(\tau)\big)\big|^2\lesssim \int_\tau^\infty \lambda^{-2}(\sigma)m\big(\xi(\sigma)\big)\omega^{-1}(\sigma)\big|h\big(\sigma,\xi(\sigma)\big)\big|\,d\sigma\cdot\int_\tau^\infty \omega(\sigma)m^{-1}\big(\xi(\sigma)\big)\,d\sigma.$$

The second integral has size $O(1)$, therefore

$$\lambda^{-2}(\tau)\big|z\big(\tau,\xi(\tau)\big)\big|^2\lesssim \int_\tau^\infty \lambda^{-2}(\sigma)m\big(\xi(\sigma)\big)\omega^{-1}(\sigma)\big|h\big(\sigma,\xi(\sigma)\big)\big|\,d\sigma. \tag{6.29}$$

Hence integrating with respect to ξ to obtain

$$\left\|z(\tau)\right\|_{L^2}^2\lesssim \int_\tau^\infty\int_0^\infty m(\xi)\omega^{-1}(\sigma)\big|h(\sigma,\xi)\big|^2\,d\xi\,d\sigma.$$

This directly implies that

$$\tau^{2(N-1)}\left\|z(\tau)\right\|_{L^2}^2\lesssim \|h\|^2_{l^2_{N-\frac{1}{\beta+1}}L^2_m}$$

which gives (6.27).

Finally, for (6.28), from (6.29) by using again Cauchy–Schwarz we obtain

$$\int_0^\infty \omega(\tau)\lambda^{-2}(\tau)\big|z\big(\tau,\xi(\tau)\big)\big|^2\,d\tau\lesssim\int_0^\infty \lambda^{-2}(\sigma)m\big(\xi(\sigma)\big)\omega^{-1}(\sigma)\big|h\big(\sigma,\xi(\sigma)\big)\big|\,d\sigma$$

and integrating with respect to ξ,

$$\int_0^\infty \omega(\tau)\|z(\tau)\|_{L^2_{1/m}}^2\,d\tau \lesssim \int_0^\infty \omega^{-1}(\sigma)\|h(\sigma)\|_{L^2_m}^2\,d\sigma.$$

Since Eq. (6.19) is solved backward in τ, we can add any nondecreasing weight in the above estimate. In particular we obtain

$$\int_{\tau_1}^{2\tau_1} \tau^{2(N-1)-\frac{\beta}{\beta+1}}\|z(\tau)\|_{L^2_{1/m}}^2\,d\tau \lesssim \int_{\tau_1}^\infty \min\{\sigma,2\tau_1\}^{2(N-1)+\frac{\beta}{\beta+1}}\|h(\sigma)\|_{L^2_m}^2\,d\sigma.$$

Hence (6.28) follows. $\square$

It remains to show that the right-hand side terms in (6.8) are perturbative. We solve Eq. (6.8) iteratively and seek a solution as the sum of the series

$$\begin{bmatrix} x_0 \\ x \end{bmatrix} = \left(\sum_{k=0}^\infty (TQ)^k\right) T \begin{bmatrix} g_0 \\ g \end{bmatrix}. \tag{6.30}$$

It remains to establish the convergence of the above series. By Lemmas 6.3, 6.4 the backward solution operator T for P, given by

$$T = \begin{bmatrix} T_{\mathrm{e}} & 0 \\ 0 & T_{\mathrm{c}} \end{bmatrix},$$

is bounded

$$T : Y_0^N \times Y^N \to X_0^N \times X^N.$$

Hence an easy way to establish the convergence of the series in (6.30) would be to show that

$$\|Q\|_{X_0^N\times X^N \to Y_0^N \times Y^N} \ll 1.$$

We can establish such a bound for certain components of Q, but as a whole Q is not even bounded in the above setting. Lacking this, a weaker but still sufficient alternative would be to prove that

$$\|TQ\|_{X_0^N\times X^N \to X_0^N \times X^N} < 1.$$

This is still not true, but we will establish a weaker bound, namely

$$\|TQ\|_{X_0^N\times X^N \to X_0^N \times X^N} \lesssim 1. \tag{6.31}$$

This ensures that all the terms in the series in (6.30) belong to $X_0^N \times X^N$. In order to ensure convergence we will split Q into two parts,

$$Q = Q_{\mathrm{g}} + Q_{\mathrm{b}}.$$

The good component Q_{g} contains most of Q and satisfies a favorable bound

$$\|TQ_{\mathrm{g}}\|_{X_0^N\times X^N\to X_0^N\times X^N}\lesssim \frac{1}{N}+\tau_0^{-\delta},\quad \delta>0. \tag{6.32}$$

Here the constant on the right can be made arbitrarily small by choosing N and τ_0 large enough. For the single bad component Q_{b} of Q we cannot establish outright smallness. However, we will show that for a large enough n we have

$$\left\|(TQ_{\mathrm{b}})^n\right\|_{X_0^N\times X^N\to X_0^N\times X^N}\ll 1. \tag{6.33}$$

Combining (6.32) and (6.33) it follows that for large enough N and τ_0 we have

$$\left\|(TQ)^n\right\|_{X_0^N\times X^N\to X_0^N\times X^N}\ll 1.$$

This ensures the convergence of the series in (6.30) in $X_0^N\times X^N$. Given the bounds in Lemmas 6.3, 6.4, the proof of Proposition 6.2 is concluded. It remains to show that Q admits a decomposition which satisfies (6.32) and (6.33).

We begin with the easiest part, namely

$$Q_{\mathrm{ce}}x_0=\omega^2\mathcal{R}_{\mathrm{ce}}x_0-2\omega\mathcal{K}_{\mathrm{ce}}\partial_\tau x_0$$

which will be included in Q_{g}. Since the kernel K_{ce} is bounded and rapidly decreasing at infinity we obtain

$$\|2\omega\mathcal{K}_{\mathrm{ce}}\partial_\tau x_0\|_{l^\infty_{N-\frac{1}{2(\beta+1)}}L^2_{\rho m}}\lesssim\|\partial_\tau x_0\|_{L^\infty_{N-\frac{\beta}{\beta+1}}}$$

which yields a $\tau^{\frac{1}{2(\beta+1)}}$ gain,

$$\|\omega\mathcal{K}_{\mathrm{ce}}\partial_\tau x_0\|_{Y^{N+\frac{1}{2(\beta+1)}}}\lesssim\|x_0\|_{X_0^N}. \tag{6.34}$$

For the second part $\omega^2\mathcal{R}_{\mathrm{ce}}x_0$ of $Q_{\mathrm{ce}}x_0$ such a simple bound no longer suffices, and we need to use some cancellations. The final result is somewhat similar to the one above, in that it gains a power of τ provided that $\beta>3/2$.

Lemma 6.5. *The following estimate holds:*

$$\left\|T\omega^2\mathcal{R}_{\mathrm{ce}}x_0\right\|_{X^{N+\frac{2\beta-3}{2(\beta+1)}}}\lesssim\|x_0\|_{X_0^N}. \tag{6.35}$$

Proof. Suppose that

$$\|x_0\|_{X_0^N}=1.$$

We denote $g = \omega^2 \mathcal{R}_{\mathrm{ce}} x_0$ and $x = T_{\mathrm{c}} g$. As before we also introduce the auxiliary variables $y = \rho^{\frac{1}{2}} x$ and $h = \rho^{\frac{1}{2}} g$. The kernel R_{ce} of $\mathcal{R}_{\mathrm{ce}}$ is bounded, rapidly decreasing at infinity and has symbol-like behavior at both 0 and infinity. Then for the function h we directly estimate

$$\left|\tau^{N-\frac{1}{\beta+1}}(1+\xi)h(\tau)\right| + \left|\tau^{N+\frac{\beta-1}{\beta+1}}(1+\xi)(\partial_\tau - 2\omega\xi\partial_\xi)h(\tau)\right| \lesssim \|x_0\|_{X_0^N} = 1.$$

As in the proof of Lemma 6.4 we have

$$y\big(\tau,\xi(\tau)\big) = \int_\tau^\infty U\big(\tau,\sigma,\xi(\tau)\big)h\big(\sigma,\xi(\sigma)\big)\,d\sigma,$$

where

$$U(\tau,\sigma,\xi) = \xi(\tau)^{-\frac{1}{2}}\frac{\lambda(\tau)}{\lambda(\sigma)}\sin\left(\xi(\tau)^{\frac{1}{2}}\lambda(\tau)\int_\tau^\sigma \lambda^{-1}(u)\,du\right).$$

Hence for y we use (6.21) and (6.22) to obtain the pointwise bound

$$\left|y\big(\tau,\xi(\tau)\big)\right| \lesssim \xi(\tau)^{-\frac{1}{2}}\min\left\{1,\xi(\tau)^{\frac{1}{2}}\omega(\tau)^{-1}\right\}\int_\tau^\infty \frac{\lambda(\tau)}{\lambda(\sigma)}\sigma^{-N+\frac{1}{\beta+1}}\big(1+\xi(\sigma)\big)^{-1}\,d\sigma$$

which we rewrite in the form

$$\omega(\tau)\left|y(\tau,\xi)\right| \lesssim \xi^{-\frac{1}{2}}\min\left\{1,\xi^{\frac{1}{2}}\omega(\tau)^{-1}\right\}\int_\tau^\infty \frac{\lambda(\tau)}{\lambda(\sigma)}\frac{\sigma^{-N-\frac{\beta-1}{\beta+1}}}{(1+\xi\lambda^2(\tau)\lambda^{-2}(\sigma))}\,d\sigma. \tag{6.36}$$

To bound $\xi^{\frac{1}{2}}y$ we integrate by parts,

$$\begin{aligned} y\big(\tau,\xi(\tau)\big) &= \xi(\tau)^{-1}\int_\tau^\infty h\big(\sigma,\xi(\sigma)\big)\partial_\sigma\left(1-\cos\left(\xi(\tau)^{\frac{1}{2}}\lambda(\tau)\int_\tau^\sigma \lambda^{-1}(u)\,du\right)\right)d\sigma \\ &= \xi(\tau)^{-1}\int_\tau^\infty \left(\cos\left(\xi(\tau)^{\frac{1}{2}}\lambda(\tau)\int_\tau^\sigma \lambda^{-1}(u)\,du\right)-1\right)\partial_\sigma h\big(\sigma,\xi(\sigma)\big)\,d\sigma. \end{aligned}$$

Estimating either $|1-\cos v| \lesssim 1$ or $|1-\cos v| \lesssim |v|$ this leads to a bound which is weaker than (6.36), namely

$$\xi^{\frac{1}{2}}\left|y(\tau,\xi)\right| \lesssim \xi^{-\frac{1}{2}}\min\left\{1,\xi^{\frac{1}{2}}\omega(\tau)^{-1}\right\}\int_\tau^\infty \frac{\sigma^{-N-\frac{\beta-1}{\beta+1}}}{(1+\xi\lambda^2(\tau)\lambda^{-2}(\sigma))}\,d\sigma. \tag{6.37}$$

In a similar manner we evaluate $D_\tau y$,

$$D_\tau y\big(\tau,\xi(\tau)\big)=\int_\tau^\infty \frac{\lambda(\tau)}{\lambda(\sigma)}h\big(\sigma,\xi(\sigma)\big)\cos\left(\xi(\tau)^{\frac12}\lambda(\tau)\int_\tau^\sigma \lambda^{-1}(u)\,du\right)d\sigma$$
$$=\xi(\tau)^{-\frac12}\int_\tau^\infty \sin\left(\xi(\tau)^{\frac12}\lambda(\tau)\int_\tau^\sigma \lambda^{-1}(u)\,du\right)\partial_\sigma h\big(\sigma,\xi(\sigma)\big)\,d\sigma$$

which leads to the same bound as in (6.37). Summing up, for z as in (6.23) we obtain

$$\big|z(\tau,\xi)\big|\lesssim \xi^{-\frac12}\min\big\{1,\xi^{\frac12}\omega(\tau)^{-1}\big\}\int_\tau^\infty \frac{\sigma^{-N-\frac{\beta-1}{\beta+1}}}{(1+\xi\lambda^2(\tau)\lambda^{-2}(\sigma))}\,d\sigma. \tag{6.38}$$

It remains to evaluate the integral on the right. If $\xi<2$ then we can neglect the first factor in the denominator of the integrand and evaluate

$$\int_\tau^\infty \frac{\sigma^{-N-\frac{\beta-1}{\beta+1}}}{(1+\xi\lambda^2(\tau)\lambda^{-2}(\sigma))}\,d\sigma\lesssim \tau^{-N+\frac{2}{\beta+1}},\quad \xi\leqslant 2.$$

However, if $\xi>2$ then this factor yields rapid decay when

$$\xi\lambda^2(\tau)\lambda^{-2}(\sigma)>1$$

which corresponds to

$$\sigma\lesssim \tau+(\log\xi)^{\beta+1}.$$

Thus we obtain

$$\int_\tau^\infty \frac{\sigma^{-N-\frac{\beta-1}{\beta+1}}}{(1+\xi\lambda^2(\tau)\lambda^{-2}(\sigma))}\,d\sigma\lesssim \big(\tau+(\log\xi)^{\beta+1}\big)^{-N+\frac{2}{\beta+1}},\quad \xi\geqslant 2.$$

Summing up, for z we have obtained the pointwise bound

$$\big|z(\tau,\xi)\big|\lesssim\begin{cases}\tau^{-N+\frac{\beta+2}{\beta+1}}, & \xi<\omega^2(\tau),\\ \xi^{-\frac12}\tau^{-N+\frac{2}{\beta+1}}, & \omega^2(\tau)\leqslant\xi\leqslant 2,\\ \xi^{-\frac12}(\tau+(\log\xi)^{\beta+1})^{-N+\frac{2}{\beta+1}}, & \xi\geqslant 2.\end{cases}$$

This allows us to estimate L^2 norms, namely

$$\big\|z(\tau,\xi)\big\|_{L^2_{1/m}}\lesssim \tau^{-N+\frac{2}{\beta+1}},$$

respectively

$$\big\|z(\tau,\xi)\big\|_{L^2}\lesssim \tau^{-N+\frac{5}{2(\beta+1)}}.$$

Finally we obtain

$$\|z\|_{L^\infty_{N-\frac{5}{2(\beta+1)}}L^2 \cap l^\infty_{N-\frac{5}{2(\beta+1)}}L^2_{1/m}} \lesssim 1$$

and the conclusion of the lemma follows. □

Next, we turn to the term $\mathcal{Q}_{\mathrm{ec}}x$ given by

$$\mathcal{Q}_{\mathrm{ec}}x = \omega^2 \mathcal{R}_{\mathrm{ec}}x - 2\omega \mathcal{K}_{\mathrm{ec}} D_\tau x.$$

We will prove that $\mathcal{Q}_{\mathrm{ec}}x$ can also be included in Q_{g}. The kernel $R_{\mathrm{ec}}(\xi)$ of $\mathcal{R}_{\mathrm{ec}}$ is bounded and decays rapidly at infinity. Then the contribution of the first term is easy to estimate using the $L^\infty_N L^2_\rho$ type bounds on x and $\xi^{\frac{1}{2}}x$,

$$\left\|\omega^2 R_{\mathrm{ec}}x\right\|_{Y_0^{N+\frac{\beta-1}{\beta+1}-\delta}} \lesssim \|x\|_{X^N}, \quad \delta > 0, \tag{6.39}$$

with δ arbitrarily small. The bound for the second term in $\mathcal{Q}_{\mathrm{ec}}x$ is similar:

Lemma 6.6. *For $\delta > 0$ we have*

$$\|T_0 \omega \mathcal{K}_{\mathrm{ec}} D_\tau x\|_{X_0^{N+\frac{\beta-1}{\beta+1}-\delta}} \lesssim \|x\|_{X^N}. \tag{6.40}$$

Proof. Set $y = \rho^{\frac{1}{2}}x$. The solution $x_0 = T_0\omega\mathcal{K}_{\mathrm{ec}}x$ is represented as

$$\begin{aligned} x_0(\tau) &= \int_\tau^\infty \int_0^\infty U_0(\tau,\sigma)\omega K_{\mathrm{ec}}(\xi) D_\sigma x(\sigma,\xi)\, d\xi\, d\sigma \\ &= \int_\tau^\infty \int_0^\infty U_0(\tau,\sigma)\omega K_{\mathrm{ec}}(\xi)\big(\partial_\sigma - 2\omega(\xi\partial_\xi + 1)\big) y(\sigma,\xi)\, d\xi\, d\sigma. \end{aligned}$$

Integrating by parts we obtain

$$x_0(\tau) = \int_\tau^\infty \int_0^\infty -\partial_\sigma U_0(\tau,\sigma)\omega(\sigma)K_{\mathrm{ec}}(\xi)y(\sigma,\xi) + 2U_0(\tau,\sigma)\omega^2(\sigma)y(\sigma,\xi)\xi\partial_\xi K_{\mathrm{ec}}(\xi)\, d\xi\, d\sigma.$$

Hence

$$\left|x_0(\tau)\right| \leqslant \int_\tau^\infty \int_0^\infty \omega(\sigma)(1+\xi)^{-1}\left|x(\sigma,\xi)\right| d\xi\, d\sigma$$

therefore

$$\left\|\tau^{N-1-\frac{1}{\beta+1}-\delta}x_0\right\|_{L^2} \lesssim \|x\|_{X^N}.$$

A similar computation yields

$$\left|\partial_\tau x_0(\tau)\right| \leqslant \int_\tau^\infty\int_0^\infty \omega^2(\sigma)\left(\tau^{-\frac{1}{\beta+1}} + \frac{\lambda(\tau)}{\lambda(\sigma)}\right)(1+\xi)^{-1}\left|x(\sigma,\xi)\right| d\xi\, d\sigma$$

which leads to

$$\left\|\tau^{N-\frac{1}{\beta+1}-\delta}\partial_\tau x_0\right\|_{L^2} \lesssim \|x\|_{X^N}.$$

The desired conclusion follows. $\square$

Finally we consider the expression $Q_{\mathrm{cc}}x$ which has the form

$$Q_{\mathrm{cc}}x = \omega^2\mathcal{R}_{\mathrm{cc}}x - 2\omega\mathcal{K}_0 D_\tau. \tag{6.41}$$

The first term is better behaved and can be included in Q_{g}:

Lemma 6.7. *For $\delta > 0$ we have the following bound:*

$$\left\|\omega^2\mathcal{R}_{\mathrm{cc}}x\right\|_{Y^{N+\frac{\beta-1}{\beta+1}-\delta}} \lesssim \|x\|_{X^N}. \tag{6.42}$$

Proof. By the definition of the X^N and Y^N spaces, it suffices to show that

$$\|\mathcal{R}_{\mathrm{cc}}x\|_{L^2_\rho} \lesssim \tau^\delta\left(\left\|\xi^{\frac{1}{2}}x\right\|_{L^2_\rho} + \tau^{-\frac{\beta}{\beta+1}}\|x\|_{L^2_\rho}\right).$$

This in turn follows by duality and dyadic summation from the bound

$$\left\|\chi_{[0,h)}\mathcal{R}^*_{\mathrm{cc}}f\right\|_{L^2_\rho} \lesssim \min\left(h^{\frac{1}{2}-\delta}, 1\right)\|f\|_{L^2_\rho}. \tag{6.43}$$

For this we need to prove that the operator $\mathcal{R}^*_{\mathrm{cc}}$ is quasi-smoothing according to the following definition:

Definition 6.8. A bounded operator $T: L^2_\rho(\mathbb{R}^+) \to L^2_\rho(\mathbb{R}^+)$ is *quasi-smoothing* if for each $\delta > 0$ there exists $C_\delta > 0$ so that

$$\|\chi_{[0,h)}\,Tf\|_{L^2_\rho} \leqslant C_\delta \min\left(h^{\frac{1}{2}-\delta}, 1\right)\|f\|_{L^2_\rho} \tag{6.44}$$

for all $h > 0$.

We remark that the quasi-smoothing operators form an ideal under composition from the right within the algebra of bounded operators. Hence, given the expression of $\mathcal{R}_{\mathrm{cc}}$, it suffices to show that the following operators are quasi-smoothing:

$$\mathcal{K}_0, \quad [\xi\partial_\xi, \mathcal{K}_0], \quad \mathcal{K}_{\mathrm{ce}}\mathcal{K}_{\mathrm{ec}}, \quad J_{\mathrm{cc}}.$$

Recall that

$$\mathcal{K}_0 f(\xi) = \int_0^\infty \frac{\rho(\xi)F(\xi,\eta)}{\xi-\eta} f(\eta)\,d\eta$$

where

$$\left|F(\xi,\eta)\right| \lesssim \min\left[\xi+\eta, (\xi+\eta)^{-\frac{5}{2}}\left(1+\left|\xi^{\frac{1}{2}}-\eta^{\frac{1}{2}}\right|\right)^{-N}\right].$$

Let $F_1(\xi,\eta) := \rho(\xi)F(\xi,\eta)$. Then

$$\mathcal{K}_0 f(\xi) = \int_0^\infty \frac{F_1(\xi,\eta)}{\xi-\eta}\chi_{[\frac{\xi}{\eta}\in[\frac{1}{2},2]]} f(\eta)\,d\eta + \int_0^\infty \frac{F_1(\xi,\eta)}{\xi-\eta}\chi_{[\frac{\xi}{\eta}\notin[\frac{1}{2},2]]} f(\eta)\,d\eta.$$

For the first operator on the right-hand side one has

$$\left|F_1(\xi,\eta)\right|\chi_{[\frac{\xi}{\eta}\in[\frac{1}{2},2]]} \leqslant \min(\xi,1) \tag{6.45}$$

which implies that the corresponding operator is quasi-smoothing, see the proof of L^2_ρ boundedness of $\mathcal{K}_0$ in the previous section. For the second operator, observe that

$$\frac{|F_1(\xi,\eta)|}{|\xi-\eta|}\chi_{[\frac{\xi}{\eta}\notin[\frac{1}{2},2]]} \lesssim \min\left(1, (\xi+\eta)^{-N}\right)$$

by the rapid off-diagonal decay of F. Hence,

$$\sup_{\xi\geqslant 0}\left|\int_0^\infty \frac{F_1(\xi,\eta)}{\xi-\eta}\chi_{[\frac{\xi}{\eta}\notin[\frac{1}{2},2]]} f(\eta)\,d\eta\right| \lesssim \|f\|_{L^2_\rho}$$

and therefore

$$\left\|\chi_{[0,h)}\int_0^\infty \frac{F_1(\xi,\eta)}{\xi-\eta}\chi_{[\frac{\xi}{\eta}\notin[\frac{1}{2},2]]} f(\eta)\,d\eta\right\|_{L^2} \lesssim h^{\frac{1}{2}}\|f\|_{L^2_\rho}$$

as desired.

For the commutator $[\xi\partial_\xi, \mathcal{K}_0]$ we have

$$[\xi\partial_\xi, \mathcal{K}_0] f(\xi) = \int_0^\infty \frac{(\xi\partial_\xi + \eta\partial_\eta) F_1(\xi,\eta)}{\xi - \eta} f(\eta)\, d\eta$$

and one argues as before.

The kernel of operator $\mathcal{K}_{\mathrm{ce}}\mathcal{K}_{\mathrm{ec}}$ is $\rho(\xi)K_{\mathrm{e}}(\xi)K_{\mathrm{e}}(\eta)$, and is bounded by $(1+\xi)^{-n}(1+\eta)^{-n}$. The quasi-smoothing property easily follows. Finally $\mathcal{J}_{\mathrm{cc}}$ is quasi-smoothing due to the kernel bound (6.7). $\square$

It remains to consider the second part of $\mathcal{Q}_{\mathrm{cc}}$ namely the expression $\omega\mathcal{K}_0 D_\tau x$. Since the kernel for $\mathcal{K}_0$ decays at 0 and at infinity, it is easy to establish the bound

$$\|\mathcal{K}_0\|_{L^2_{\rho/m} \to L^2_{\rho m}} \lesssim 1.$$

It follows that

$$\|\omega\mathcal{K}_0 D_\tau x\|_{Y^N} \lesssim \|x\|_{X^N}. \tag{6.46}$$

The difficulty is that there is no smallness in the above relation, and it is not possible to gain any smallness by letting τ be large enough. To deal with this we reiterate:

Lemma 6.9. *Suppose that n is large enough. Then*

$$\left\|(T\omega\mathcal{K}_0 D_\tau)^n x\right\|_{X^N} \ll \|x\|_{X^N}. \tag{6.47}$$

Proof. By (6.46) and Lemma 6.4 it suffices to prove that for large enough n,

$$\left\|(\omega\mathcal{K}_0 D_\tau T)^n g\right\|_{l^\infty_N L^2_{\rho m}} \ll \|g\|_{l^\infty_N L^2_{\rho m}}. \tag{6.48}$$

We divide the operator $\mathcal{K}_0$ in two parts,

$$\mathcal{K}_0 = \mathcal{K}_0^{\mathrm{d}} + \mathcal{K}_0^{\mathrm{nd}}$$

with kernels

$$K_0^{\mathrm{d}}(\xi,\eta) = K_0(\xi,\eta)\chi_{[|\frac{\xi}{\eta}-1|<\frac{1}{n}]}, \qquad K_0^{\mathrm{nd}}(\xi,\eta) = K_0(\xi,\eta)\chi_{[|\frac{\xi}{\eta}-1|>\frac{1}{n}]}.$$

The contribution of $\mathcal{K}_0^{\mathrm{nd}}$ is non-resonant, we and we expect to gain powers of τ from oscillations. Precisely, we will prove that

$$\left\|(\omega\mathcal{K}_0 D_\tau T)\big(\omega\mathcal{K}_0^{\mathrm{nd}} D_\tau T\big) g\right\|_{l^\infty_{N+\frac{\beta-2}{\beta+1}} L^2_{\rho m}} \lesssim_n \|g\|_{l^\infty_N L^2_{\rho m}}. \tag{6.49}$$

Here the implicit constant depends on n, but that is not important since we gain a power of τ.

Assuming (6.49) holds, in order to prove (6.48) it remains to show that for large n we have

$$\left\| \left(\omega \mathcal{K}_0^{\mathrm{d}} D_\tau T\right)^n g \right\|_{l_N^\infty L^2_{\rho m}} \ll \|g\|_{l_N^\infty L^2_{\rho m}}. \tag{6.50}$$

Proof of (6.50). For another small parameter ε to be chosen later we further divide $\mathcal{K}_0^{\mathrm{d}}$ into three parts,

$$\mathcal{K}_0^{\mathrm{d}} = \mathcal{K}_{0,1}^{\mathrm{d},\varepsilon} + \mathcal{K}_{0,2}^{\mathrm{d},\varepsilon} + \mathcal{K}_{0,3}^{\mathrm{d},\varepsilon}$$

with kernels

$$K_{0,1}^{\mathrm{d},\varepsilon}(\xi,\eta) = 1_{\xi<\varepsilon} K_0^{\mathrm{d}}(\xi,\eta), \qquad K_{0,3}^{\mathrm{d},\varepsilon}(\xi,\eta) = 1_{\xi>\varepsilon^{-1}} K_0^{\mathrm{d}}(\xi,\eta).$$

The center part $\mathcal{K}_{0,2}^{\mathrm{d},\varepsilon}$ enjoys better localization, while the two tails $\mathcal{K}_{0,1}^{\mathrm{d},\varepsilon}$ and $\mathcal{K}_{0,3}^{\mathrm{d},\varepsilon}$ are small. Precisely,

$$\left\| \omega \mathcal{K}_{0,1}^{\mathrm{d},\varepsilon} D_\tau T g \right\|_{l_N^\infty L^2_{\rho m}} + \left\| \omega \mathcal{K}_{0,3}^{\mathrm{d},\varepsilon} D_\tau T g \right\|_{l_N^\infty L^2_{\rho m}} \lesssim \varepsilon^{\frac{1}{4}} \|g\|_{l_N^\infty L^2_{\rho m}}. \tag{6.51}$$

It is easy to see that due to the supports of the kernels we have

$$\mathcal{K}_{0,1}^{\mathrm{d},\varepsilon} D_\tau T \mathcal{K}_{0,2}^{\mathrm{d},\varepsilon(1+\frac{1}{n})} = 0$$

and

$$\mathcal{K}_{0,2}^{\mathrm{d},\varepsilon(1+\frac{1}{n})} D_\tau T \mathcal{K}_{0,3}^{\mathrm{d},\varepsilon(1+\frac{1}{n})} = 0.$$

Hence we obtain the decomposition

$$\begin{aligned}
\left(\omega \mathcal{K}_0^{\mathrm{d}} D_\tau T\right)^n = {} & \left(\omega \mathcal{K}_{0,2}^{\mathrm{d},\varepsilon} D_\tau T\right)^n + \sum_{k=1}^{n} \left(\omega \mathcal{K}_{0,2}^{\mathrm{d},\varepsilon} D_\tau T\right)^{k-1} \left(\omega \mathcal{K}_{0,1}^{\mathrm{d},\varepsilon} D_\tau T\right) \left(\omega \mathcal{K}_{0,1}^{\mathrm{d},2\varepsilon} D_\tau T\right)^{n-k} \\
& + \sum_{j=1}^{n} \left(\omega \mathcal{K}_{0,3}^{\mathrm{d},2\varepsilon} D_\tau T\right)^{j-1} \left(\omega \mathcal{K}_{0,3}^{\mathrm{d},\varepsilon} D_\tau T\right) \left(\omega \mathcal{K}_{0,2}^{\mathrm{d},\varepsilon} D_\tau T\right)^{n-j} \\
& + \sum_{1\leqslant j<k\leqslant n} \left(\omega \mathcal{K}_{0,3}^{\mathrm{d},2\varepsilon} D_\tau T\right)^{j-1} \left(\omega \mathcal{K}_{0,3}^{\mathrm{d},\varepsilon} D_\tau T\right) \left(\omega \mathcal{K}_{0,2}^{\mathrm{d},\varepsilon} D_\tau T\right)^{k-j-1} \left(\omega \mathcal{K}_{0,1}^{\mathrm{d},\varepsilon} D_\tau T\right) \\
& \times \left(\omega \mathcal{K}_{0,1}^{\mathrm{d},2\varepsilon} D_\tau T\right)^{n-k}.
\end{aligned}$$

For the middle part we will prove the bound

$$\left\| \left(\omega \mathcal{K}_{0,2}^{\mathrm{d},\varepsilon} D_\tau T\right)^k g \right\|_{l_N^\infty L^2_{\rho m}} \leqslant \frac{(C|\log \varepsilon|)^k}{(k-1)!} \|g\|_{l_N^\infty L^2_{\rho m}}. \tag{6.52}$$

Combining this with (6.51) we obtain

$$\left\| \left(\omega \mathcal{K}_0^{\mathrm{d}} D_\tau T\right)^n g \right\|_{l_N^\infty L^2_{\rho m}} \leqslant \left(\sum_{k=0}^{n} \frac{(C|\log \varepsilon|)^k}{(k-1)!} \varepsilon^{\frac{n-k}{4}} \right) \|g\|_{l_N^\infty L^2_{\rho m}}.$$

Choosing $\varepsilon = n^{-4}$ this gives

$$\left\| \left(\omega \mathcal{K}_0^{\mathrm{d}} D_\tau T\right)^n g \right\|_{l_N^\infty L^2_{\rho m}} \leqslant \frac{(C \log n)^n}{n^{n-2}} \|g\|_{l_N^\infty L^2_{\rho m}}$$

for a new constant C. Thus (6.50) is established for n sufficiently large.

We return to prove (6.52). Since

$$D_\tau U(\tau, \sigma, \xi) := V(\tau, \sigma, \xi) = \frac{\lambda(\tau)}{\lambda(\sigma)} \cos\left(\xi^{\frac{1}{2}} \lambda(\tau) \int_\tau^\sigma \lambda^{-1}(\tau) \right)$$

we can write the function

$$y(\tau, \xi) = \rho(\xi)^{\frac{1}{2}} \left(\omega \mathcal{K}_{0,2}^{\mathrm{d},\varepsilon} D_\tau T\right)^n g$$

in the form

$$\begin{aligned} y(\tau, \xi) = {} & \int_\tau^\infty \int_{\mathbb{R}^+} d\sigma_1 \, d\eta_0 \, \omega \mathcal{K}_{0,2}^{\mathrm{d},\varepsilon}(\xi, \eta_0) V(\tau, \sigma_1, \eta_0) \\ & \times \int_{\sigma_1}^\infty \int_{\mathbb{R}^+} d\sigma_2 \, d\eta_1 \, \omega \mathcal{K}_{0,2}^{\mathrm{d},\varepsilon}\left(\eta_0 \frac{\lambda^2(\sigma_0)}{\lambda^2(\sigma_1)}, \eta_1 \right) V(\sigma_1, \sigma_2, \eta_1) \ldots \\ & \times \int_{\sigma_{n-1}}^\infty \int_{\mathbb{R}^+} d\sigma_n \, d\eta_n \, \omega \mathcal{K}_{0,2}^{\mathrm{d},\varepsilon}\left(\eta_{n-2} \frac{\lambda^2(\sigma_{n-2})}{\lambda^2(\sigma_{n-1})}, \eta_{n-1} \right) V(\sigma_{n-1}, \sigma_n, \eta_{n-1}) \\ & \times \int_{\mathbb{R}^+} d\eta_n \, \omega \mathcal{K}_{0,2}^{\mathrm{d},\varepsilon}\left(\eta_{n-1} \frac{\lambda^2(\sigma_{n-1})}{\lambda^2(\sigma_n)}, \eta_n \right) h(\sigma_n, \eta_n) \frac{\lambda(\tau)}{\lambda(\sigma_m)}. \end{aligned}$$

In order for the above integrand to be nonzero we must have

$$\left| \frac{\eta_k \lambda^2(\sigma_k)}{\eta_{k+1} \lambda^2(\sigma_{k+1})} - 1 \right| \leqslant \frac{1}{n}, \qquad \left| \frac{\xi}{\eta_0} - 1 \right| \leqslant \frac{1}{n}.$$

This implies that

$$\eta_n \lambda^2(\sigma_n) \leqslant 3\xi \lambda^2(\tau).$$

Since $\varepsilon < \sigma_n, \xi \leqslant \varepsilon^{-1}$ it follows that

$$\lambda^2(\sigma_n) \leqslant 3\varepsilon^2\lambda^2(\tau).$$

If τ is sufficiently large this implies that

$$\sigma_n \leqslant \sigma(\tau) = \tau + C\tau^{\frac{\beta}{\beta+1}}|\log\varepsilon|^{\beta+1}.$$

Using the L^2 boundedness of $\mathcal{K}_{0,2}^{\mathrm{d},\varepsilon}$ and of the transport along the flow (as $|V| \leqslant 1$) it follows that

$$\|y(\tau)\|_{L^2_{\rho m}} \leqslant m^2(\varepsilon)\big(C\omega(\tau)\big)^{n+1}\int_\tau^{\sigma(\tau)} d\sigma_1 \int_{\sigma_1}^{\sigma(\tau)} d\sigma_2 \ldots \int_{\sigma_{n-1}}^{\sigma(\tau)} \|h(\sigma_n)\|_{L^2_{\rho/m}}\, d\sigma_n.$$

Changing the order of integration this yields

$$\|y(\tau)\|_{L^2_{\rho m}} \leqslant m^2(\varepsilon)\frac{(C\omega(\tau))^n}{(n-1)!}\int_\tau^{\sigma(\tau)} (\tau-\sigma_n)^{n-1}\|h(\sigma_n)\|_{L^2_{\rho/m}}\, d\sigma_n.$$

Since

$$\int_\tau^{\sigma(\tau)} (\tau-\sigma_n)^{n-1}\, d\sigma^n \approx \frac{1}{n}\omega(\tau)^{-n}$$

we finally obtain

$$\|y\|_{l^\infty_{N+\frac{\beta}{\beta+1}} L^2_{\rho m}} \leqslant m^2(\varepsilon)\frac{(C|\log\varepsilon|^{\beta+1})^n}{n!}\|h\|_{l^\infty_N L^2_{\rho/m}}.$$

Thus (6.52) is proved.

Proof of (6.49). Denoting $x = T(\omega\mathcal{K}_0^{\mathrm{nd}}D_\tau T)g$, $y = \rho^{\frac{1}{2}}x$ and $h = \rho^{\frac{1}{2}}g$ we need to prove that

$$\|D_\tau y\|_{l^\infty_{N-\frac{2}{\beta+1}} L^2_{1/m}} \lesssim \|h\|_{l^\infty_N L^2_m}. \tag{6.53}$$

Due to the formula (6.20) we have the integral representation

$$D_\tau y(\tau,\xi) = \int_\tau^\infty \omega(s)\frac{\lambda(\tau)}{\lambda(s)}\cos\left(\xi^{\frac{1}{2}}\lambda(\tau)\int_\tau^s \lambda^{-1}(\theta)\,d\theta\right)\int_0^\infty K_0^{\mathrm{nd}}\big(\xi(s),\eta(s)\big)\frac{\lambda^2(\sigma)}{\lambda^2(s)}$$
$$\times\int_s^\infty \frac{\lambda(s)}{\lambda(\sigma)}\cos\left(\eta^{\frac{1}{2}}\lambda(\sigma)\int_s^\sigma \lambda^{-1}(\theta)\,d\theta\right)y(\sigma,\eta)\,d\sigma\,d\eta\,ds$$

where $\xi(s) = \xi \frac{\lambda^2(\tau)}{\lambda^2(s)}$ and $\eta(s) = \eta \frac{\lambda^2(\sigma)}{\lambda^2(s)}$. In the support of the kernel K_0^{nd} we have $|\frac{\xi(s)}{\eta(s)} - 1| > \frac{1}{n}$ therefore $|\frac{\xi^{1/2}\lambda(\tau)}{\eta^{1/2}\lambda(\sigma)} - 1| \gtrsim \frac{1}{n}$. Thus the two oscillatory factors have different frequencies, and we can gain if we integrate by parts with respect to s. Denoting

$$u(s) = \xi^{\frac{1}{2}}\lambda(\tau)\int_\tau^s \lambda^{-1}(\theta)\,d\theta, \qquad v(s) = \eta^{\frac{1}{2}}\lambda(\sigma)\int_s^\sigma \lambda^{-1}(\theta)\,d\theta$$

we write

$$2\cos u(s)\cos v(s) = \cos\bigl(u(s)+v(s)\bigr) + \cos\bigl(u(s)-v(s)\bigr).$$

We change the order of integration in the above expression for y and integrate by parts with respect to s. Since

$$\frac{d}{ds}(u \pm v) = \lambda^{-1}(s)\bigl(\xi^{\frac{1}{2}}\lambda(\tau) \mp \eta^{\frac{1}{2}}\lambda(\sigma)\bigr) = \xi(s)^{\frac{1}{2}} \mp \eta(s)^{\frac{1}{2}}$$

we integrate the cosine and differentiate the rest to obtain

$$\begin{aligned}
D_\tau y(\tau,\xi) = {} & \sum_{\pm}\int_\tau^\infty\int_0^\infty\int_\tau^\sigma \frac{\lambda(\tau)}{\lambda(\sigma)}\frac{1}{\xi(s)^{\frac{1}{2}} \mp \eta(s)^{\frac{1}{2}}}\frac{1}{\lambda(s)}\frac{d}{ds}\left(\omega(s)K_0^{\mathrm{nd}}\bigl(\xi(s),\eta(s)\bigr)\frac{\lambda^2(\sigma)}{\lambda(s)}\right) \\
& \times \sin\bigl(u(s)+v(s)\bigr)h(\sigma,\eta)\,ds\,d\eta\,d\sigma \\
& \pm \int_\tau^\infty\int_0^\infty \omega(\tau)\frac{\lambda(\tau)}{\lambda(\sigma)}\frac{1}{\xi^{\frac{1}{2}} \mp \eta(\tau)^{\frac{1}{2}}}K_0^{\mathrm{nd}}\bigl(\xi,\eta(\tau)\bigr)\frac{\lambda^2(\sigma)}{\lambda^2(\tau)}\sin\bigl(v(\tau)\bigr)h(\sigma,\eta)\,d\eta\,d\sigma \\
& - \int_\tau^\infty\int_0^\infty \omega(\sigma)\frac{\lambda(\tau)}{\lambda(\sigma)}\frac{1}{\xi(\sigma)^{\frac{1}{2}} \mp \eta^{\frac{1}{2}}}K_0^{\mathrm{nd}}\bigl(\xi(\sigma),\eta\bigr)\sin\bigl(u(\sigma)\bigr)h(\sigma,\eta)\,d\eta\,d\sigma.
\end{aligned}$$

We have

$$\frac{d}{ds}\bigl(K_0^{\mathrm{nd}}\bigl(\xi(s),\eta(s)\bigr)\lambda^{-1}(s)\bigr) = \omega(s)(\xi\partial_\xi + \eta\partial_\eta - 1)K_0^{\mathrm{nd}}\bigl(\xi(s),\eta(s)\bigr)\lambda^{-1}(s).$$

Due to Theorem 5.1 the kernel K_0^{nd} is bounded and decays rapidly at infinity therefore we can bound it by

$$\bigl|K_0^{\mathrm{nd}}(\xi,\eta)\bigr| \lesssim_n \frac{1}{(1+\xi)(1+\eta)}.$$

We also have

$$\bigl|(\xi\partial_\xi + \eta\partial_\eta - 1)K_0^{\mathrm{nd}}(\xi,\eta)\bigr| \lesssim_n \frac{1}{(1+\xi)(1+\eta)}.$$

Hence the following rough bounds are valid:

$$\left|\frac{K_0^{\mathrm{nd}}(\xi,\eta)}{\xi^{\frac{1}{2}}\pm\eta^{\frac{1}{2}}}\right|+\left|\frac{(\xi\partial_\xi+\eta\partial_\eta-1)K_0^{\mathrm{nd}}(\xi,\eta)}{\xi^{\frac{1}{2}}\pm\eta^{\frac{1}{2}}}\right|\lesssim_n\frac{1}{\xi^{\frac{1}{2}}\eta^{\frac{1}{2}}}.$$

Inserting this in the bounds for $D_\tau y$ we obtain

$$\begin{aligned}\left|D_\tau y(\tau,\xi)\right|\lesssim_n{}&\int_\tau^\infty\int_0^\infty\int_\tau^\sigma\omega^2(s)\frac{\lambda(\tau)}{\lambda(\sigma)}\frac{1}{\xi(s)^{\frac{1}{2}}\eta(s)^{\frac{1}{2}}}\frac{\lambda^2(\sigma)}{\lambda^2(s)}\left|h(\sigma,\eta)\right|ds\,d\eta\,d\sigma\\&+\int_\tau^\infty\int_0^\infty\omega(\tau)\frac{\lambda(\tau)}{\lambda(\sigma)}\frac{1}{\xi^{\frac{1}{2}}\eta(\tau)^{\frac{1}{2}}}\frac{\lambda^2(\sigma)}{\lambda^2(\tau)}\left|h(\sigma,\eta)\right|d\eta\,d\sigma\\&+\int_\tau^\infty\int_0^\infty\omega(\sigma)\frac{\lambda(\tau)}{\lambda(\sigma)}\frac{1}{\xi(\sigma)^{\frac{1}{2}}\eta^{\frac{1}{2}}}\left|h(\sigma,\eta)\right|d\eta\,d\sigma.\end{aligned}$$

This can be rewritten in the form

$$\begin{aligned}\xi^{\frac{1}{2}}\left|D_\tau y(\tau,\xi)\right|&\lesssim_n\int_\tau^\infty\int_0^\infty\int_\tau^\sigma\omega^2(s)\frac{|h(\sigma,\eta)|}{\eta^{\frac{1}{2}}}\,ds\,d\eta\,d\sigma+\int_\tau^\infty\int_0^\infty\omega(\tau)\frac{|h(\sigma,\eta)|}{\eta^{\frac{1}{2}}}\,d\eta\,d\sigma\\&\lesssim_n\omega^2(\tau)\int_\tau^\infty\int_0^\infty\sigma\frac{|h(\sigma,\eta)|}{\eta^{\frac{1}{2}}}\,d\eta\,d\sigma.\end{aligned}$$

Taking weighted L^2 norms we obtain

$$\left\|D_\tau y(\tau,\xi)\right\|_{L^2_{1/m}}\lesssim_n\omega^2(\tau)\int_\tau^\infty\sigma\left\|h(\sigma)\right\|_{L^2_m}d\sigma.$$

Then by Cauchy–Schwarz

$$\left\|D_\tau y(\tau)\right\|_{L^2_{1/m}}\lesssim_n\tau^{-N+\frac{3}{2(\beta+1)}}\|h\|_{l_N^\infty L_m^2}$$

and further

$$\|D_\tau y\|_{l^\infty_{N-\frac{2}{\beta+1}}L^2_{1/m}}\lesssim_n\|h\|_{l_N^\infty L_m^2}.$$

Thus (6.51) is proved, and the proof of the lemma is concluded. □

Proposition 6.2 follows. □

The proof of Proposition 6.2 is also concluded. □

We now turn to the proof of Theorem 2.2 in Section 2 which estimates forward solutions ε of Eq. (2.2), which we rewrite as

$$P_0\varepsilon = f, \qquad P_0 = -\partial_t^2 + \partial_r^2 + \frac{1}{r}\partial_r + \frac{2}{r^2}\big(1 - 3Q\big(\lambda(t)r\big)^2 - 6Q\big(\lambda(t)r\big)v_{10}\big)$$

under the action of the invariant vector field $S = t\partial_t + r\partial_r$. So far we have proved the bound (2.3) for ε. In order to prove (2.4) we write an equation for $S\varepsilon$, namely

$$P_0 S\varepsilon = Sf + [P_0, S]\varepsilon.$$

A direct computation yields

$$[P_0, S] = 2P_0 - V, \qquad V = \frac{1}{r^2}S\big(3Q\big(\lambda(t)r\big)^2 - 6Q\big(\lambda(t)r\big)v_{10}\big).$$

Hence

$$P_0 S\varepsilon = (S+2)f + V\varepsilon.$$

A direct computation shows that

$$|V| \lesssim \frac{1}{r^2}\frac{R^2}{(1+R^2)^2} \lesssim \lambda(t)^2.$$

Hence applying (2.3) we obtain

$$\|S\varepsilon\|_{H^1_{N_1}} \lesssim \frac{1}{N_1}\big(\|Sf\|_{L^2_{N_1}} + \|f\|_{L^2_{N_1}} + \big\|\lambda^2\varepsilon\big\|_{L^2_{N_1}}\big).$$

Then (2.4) follows since

$$\big\|\lambda^2\varepsilon\big\|_{L^2_{N_1}} \lesssim \|\varepsilon\|_{H^1_{N_1+2\beta+1}}.$$

We remark that this requires

$$N_0 \geqslant N_1 + 2\beta + 1.$$

The proof of (2.5) is similar.

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